Mathematics in Economics – lecture 6

Extremes of a function of two real variables

Necessary condition for the extreme

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y} = 0$$

A point satisfying equalities above is called a stationary (critical) point. However, this condition is not sufficient.

In a critical point can be maximum, minimum or an inflection point. To decide which situation occurs, we use the second derivatives and a matrix called *hessian* H(*C*)

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Determinant calculation:
we multiply the numbers on the main diagonal and subtract the product of the numbers on the secondary diagonal.

Then we use Sylvester´s theorem.

We denote: D1 = $f''_{xx}(C)$ and D2 = H(*C*). Then:

If D2>0, then we have an extreme.

Moreover, If D1>0, we have a minimum, if D1<0, we have a maximum.

IF D2<0, we have an inflection point. If D2 = 0, we cannot decide.

Problem 1

Find extremes of the function $f(x,y) = x^3 - 2xy$

Solution: We start with the first derivatives:

Both derivatives must be 0, which yields the critical point C [0,0].

Now we compute all second derivatives and hessian:

We substitute point C into hessian: *Hf*(0,0) =

Because D2<0, the point C is an inflection point.

Problem 2

Find extremes of the function $f(x, y) = x^2 - 2xy + y$.

Solution: We start with the first derivatives:

Both derivatives must be 0, which yields the critical point C [1/2, 1/2].

Now we compute all second derivatives and hessian:

We substitute point C into hessian: *Hf*(1/2; 1/2) =

.

Because D2<0, the point C is an inflection point.

Problem 3

Find extremes of the function $f(x, y) = -3x^2 + 2xy - 2y^2 - 10$.

Solution: We start with the first derivatives:

Both derivatives must be 0, which yields the critical point C [0, 0].

Now we compute all second derivatives and hessian:

We substitute point C into hessian: *Hf*(0; 0) =

.

Because D2 > 0, we have extreme at the point C; because D1 < 0, we have a maximum.

Problem 4

Find extremes of the function $f(x, y) = x^2 + 4xy + 6y^2 - 2x + 8y - 5$.

Solution: We start with the first derivatives:

Both derivatives must be 0, which yields the critical point $C\,[7,\ -3]$.

Now we compute all second derivatives and hessian:

We substitute point C into hessian: *Hf*(7; −3) = .

Because D2 > 0, we have extreme at the point C; because D1 > 0, we have a minimum.

Problem 5

Find the maximum of the revenue function: $TR(Q_1, Q_2) = 50Q_1 + 20Q_2 - 2Q_1^2 - 5Q_2^2$

Solution: We start with the first derivatives:

Both derivatives must be 0, which yields the critical point $C\,[12.5\ ,\ 2]$.

Now we compute all second derivatives and hessian:

We substitute point C into hessian: *Hf*(12.5; 2) = .

Because D2 > 0, we have extreme at the point C; because D1 < 0, we have a maximum.

HOMEWORK

A] $f(x,y) = x^2 + 2y^2 - 6x + 8$

B] $f(x,y) = x^3 - xy + y$

C] $f(x,y) = 2xy - 3x^2 - 2y^2 + 10$

D] $f(x,y) = y - \frac{x^3}{3} + \ln(x - y)$