Statistical Methods for Economists

Lecture 10

Fractional Factorial Design of Experiments

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Outline of the lecture

- Fractional Factorial Design: Motivation & Introduction
- One half (½) fraction design
- One quarter (¼) fraction design
- One eight (⅛) fraction design
- For specialists in algebra: The general 2^{-p} fraction design

Fractional Factorial Design: Motivation & Introduction

Consider an observed quantity (such as a measure of quality, filtration rate in chemistry or production, the lifespan of a product, etc.); we conjecture that the observed quantity may depend upon the levels of many factors (such as the day of the week, temperature, material, and so on).

Our purpose is to carry out just a few experiments and to identify those factors that influence the observed quantity most. (In other words, we wish to identify the factors with the most dominant effect on the quantity.)

Once the effects are identified, we analyse them in greater detail then.

Fractional Factorial Design: Motivation & Introduction

Denote the observed quantity by Y and consider up to k two-level factors about which we conjecture that they might influence the value of the quantity Y.

A possibility to explore the most significant factors consists in setting up the **full factorial design of the experiment**, which is denoted by

$$2^k$$

and to run the experiment 2^k-times, i.e. once for each of the 2^k combinations of the levels of the factors.

More generally, the experiment is run $(K \times 2^k)$-times if the experiment is

Fractional Factorial Design: Motivation & Introduction

Performing the **full factorial design of the experiment,** the number of the runs may soon become infeasible because the number grows exponentially.

For $K = 1$, we have:

$k = 1 \implies 2^1 = 2$ runs	$k = 5 \implies 2^5 = 32$ runs
$k = 2 \implies 2^2 = 4$ runs	$k = 6 \implies 2^6 = 64$ runs
$k = 3 \implies 2^3 = 8$ runs	$k = 7 \implies 2^7 = 128$ runs
$k = 4 \implies 2^4 = 16$ runs	$k = 8 \implies 2^8 = 256$ runs

and so on.

Fractional Factorial Design: Motivation & Introduction

Performing the **full factorial design of the experiment**, the number of the runs may soon become infeasible due to limited resources (time, financial, material, etc.).

Moreover, such a high number (2^k) of the runs will be unnecessary (inefficient) in fact as the number k of the factors increases: The gain of the knowledge will not be adequate to the effort.

Recall that the purpose is just to identify quickly those few factors

Fractional Factorial Design: Motivation & Introduction

We actually assume the following three principles:

- **Hierarchical Ordering Principle**
  - effects of lower orders are more likely to be important than effects of higher orders; and
  - effects of the same order are equally likely to be important
- **Effect Sparsity Principle (Pareto Principle)**
  - the number of relatively important effects in a factorial experiment is small
- **Effect Heredity Principle**
  - if an interaction is significant, then at least one of its parent factors is significant

Fractional Factorial Design: Motivation & Introduction

In particular, if $k = 7$ and $K = 1$, say, then there are:

- $\binom{7}{1} = 7$ degrees of freedom for the main effects,
- $\binom{7}{2} = 21$ degrees of freedom for 2-factor interactions
- $\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 99$ d.f. for all the other interactions of higher order (i.e. interactions of orders 3, 4, 5, 6, 7)

Then, performing the full factorial design of the experiment when the number k of the factors is high, the gain of the knowledge will not be adequate to the effort put into the experiments due to the three aforementioned principles.

Fractional Factorial Design: Motivation & Introduction

Thanks to the three aforementioned principles, which we assume,

it is often sufficient to run a fraction of the original full factorial plan.

The **fractional factorial design of the experiment** is denoted by

$$2^{k-p}$$

where

- k is the number of the factors
- p is the fraction index

Moreover, since our purpose is to carry our as few experiments as possible,

we assume that the experiments are not replicated; that is, we assume

Fractional Factorial Design: Motivation & Introduction

Setting up the **fractional factorial design of the experiment**, which is denoted by

$$2^{k-p}$$

where

- k is the number of the factors
- p is the **fraction index**

the experiment is run

$$2^{k-p} = \frac{2^k}{2^p}$$

times, i.e.

once for each of the 2^{k-p} selected combinations of the levels of the factors.

Fractional Factorial Design: Motivation & Introduction

The fraction index p should satisfy the relations

$$0 \leq p \leq k \qquad \text{and} \qquad 2^{n-p} \geq k$$

$$1/2^p \geq k/2^n$$

$$2^p \leq 2^n/k$$

$$p \leq n - \log_2 k$$

- For $p = 1$, we obtain one half $(2^{-1} = {}^1\!/_2)$ fraction design.
- For $p = 2$, we obtain one quarter $(2^{-2} = {}^1\!/_4)$ fraction design.
- And so on.

One half (½) fraction design

½ fraction design

- Consider k main two-level factors in the experiment.
- For example, consider $k = 4$ factors A, B, C, D in the experiment.
- Consider first the full-factorial design; that is,
- make up a table consisting of k columns corresponding

  to the factors (such as A, B, C, D) and consisting of 2^k rows

  (such as $2^4 = 16$ rows) listing all the possible combinations

  of the signs "−" and "+" (the levels of the factors).
- The columns of the table are also called **contrasts**

  because the number of the "−" signs and "+" signs is the same in each column.

½ fraction design

- Add yet the zeroth column consisting of 2^k "+" signs only, and denote this column by "I" (the intercept term).
- The columns (such as I, A, B, C, D) of the table can be multiplied each other component-wise (i.e. the corresponding signs are multiplied together) by using the rules:

$$- - = + \qquad + - = - \qquad - + = - \qquad + + = +$$

- Notice that "I" is a neutral element:

$$IL = L = LI \qquad \text{for any column} \quad L \in \{A, B, C, D, I, \ldots\}$$

- Notice that any column is self-inverse:

Full design

We thus have

in our example

with $k = 4$ factors:

No.	A	B	C	D
1	−	−	−	−
2	+	−	−	−
3	−	+	−	−
4	+	+	−	−
5	−	−	+	−
6	+	−	+	−
7	−	+	+	−
8	+	+	+	−
9	−	−	−	+
10	+	−	−	+
11	−	+	−	+
12	+	+	−	+
13	−	−	+	+
14	+	−	+	+
15	−	+	+	+
16	+	+	+	+

½ fraction design

- We do not wish to run the full factorial experiment, i.e. to carry out the experiment for each of the 2^k combinations of the factors.
- Letting the fraction index $p = 1$, we instead wish to run ½ fraction of the full factorial design of the experiment.
- To this end, select some $k-1$ main factors (such as A, B, C if $k = 4$) and confound the remaining factor (such as D) with the interaction of the $k-1$ selected factors by introducing the equation, e.g.

$$D = ABC$$

- This equation is called the **generating equation of the alias structure.**

½ fraction design

- Consider the above table with the 2^k rows of the full factorial design.
- In the table, keep every row that fulfils the generating equation

$$D = ABC$$

- And drop every row that does not fulfil the generating equation.

- All in all, there 2^{k-1} rows remain in the table

  and the other 2^{k-1} rows have been dropped.

½ fraction design

In our example with $k = 4$ factors and with the generating equation $D = ABC$, we have:

No.	A	B	C	D = ABC
1	−	−	−	−
2	+	−	−	+
3	−	+	−	+
4	+	+	−	−
5	−	−	+	+
6	+	−	+	−
7	−	+	+	−
8	+	+	+	+

½ fraction design

- Observe that – considering the $k-1$ selected main factors (such as A, B, C in our example) – we have got a full factorial design 2^{k-1} of the experiment with respect to those $k-1$ selected main factors (such as those A, B, C).

- We then carry out the 2^{k-1} runs of the experiment with the levels of all the k factors set up according to the list in the table, that is with the levels of the factors fulfilling **the generating equation**

$$D = ABC$$

(The experiment is replicated $K = 1$ times for each combination because we wish

½ fraction design

Multiplying both sides of **the generating equation**

$$D = ABC$$

by the column-contrast "D", we obtain:

$$DD = ABCD$$

$$I = ABCD$$

This equation is called **the defining equation** of the fractional design,

and its right-hand side ("ABCD") is **the defining word** of the fractional design.

½ fraction design

The defining equation

$$I = ABCD$$

induces the following **alias structure** of the fractional factorial design 2^{k-1}:

$I = ABCD$	$A = BCD$	$AB = CD$
	$B = ACD$	$AC = BD$
	$C = ABD$	$AD = BC$
	$D = ABC$	

All in all, in our example, we have:

½ fraction design

Here the defining equation is $I = ABCD$

No.	I = ABCD	**A** = BCD	**B** = ACD	**C** = ABD	AB = CD	AC = BD	BC = AD	ABC = D
1	+	−	−	−	+	+	+	−
2	+	+	−	−	−	−	+	+
3	+	−	+	−	−	+	−	+
4	+	+	+	−	+	−	−	−
5	+	−	−	+	+	−	−	+
6	+	+	−	+	−	+	−	−
7	+	−	+	+	−	−	+	−
8	+	+	+	+	+	+	+	+

The generating equation is $D = ABC$

½ fraction design

- We then carry out the 2^{k-1} runs of the experiment with the levels of all the k factors set up according to the list in the table.

- It is, essentially, a full factorial design for the $k-1$ selected main factors (such as A, B, C in our example).

- We then evaluate the results, i.e. the significance of the main factors and the significance of their interactions, in the precisely the same way as in the case of a full factorial design – see the previous lecture.

½ fraction design

- Here, however, some factors and interactions are confounded; that is, we cannot distinguish between them.
- The respective estimate "b", which we calculate, of the coefficient "β" is always the sum of the true coefficients of the confounded factors.
- We thus have in our example:

$$I = ABCD \longrightarrow I + ABCD \qquad D = ABC \longrightarrow D + ABC$$

$$A = BCD \longrightarrow A + BCD \qquad AB = CD \longrightarrow AB + CD$$

$$B = ACD \longrightarrow B + ACD \qquad AC = BD \longrightarrow AC + BD$$

$$C = ABD \longrightarrow C + ABD \qquad AD = BC \longrightarrow AD + BC$$

½ fraction design

Here the generating equation is $D = ABC$

No.	I + ABCD	**A** + BCD	**B** + ACD	**C** + ABD	AB + CD	AC + BD	BC + AD	ABC + D
1	+	−	−	−	+	+	+	−
2	+	+	−	−	−	−	+	+
3	+	−	+	−	−	+	−	+
4	+	+	+	−	+	−	−	−
5	+	−	−	+	+	−	−	+
6	+	+	−	+	−	+	−	−
7	+	−	+	+	−	−	+	−
8	+	+	+	+	+	+	+	+

We then use the three aforementioned principles

to decide which factors and interactions are significant.

Fractional Factorial Design: The Three Principles

We assume the following three principles:

- **<u>Hierarchical Ordering</u> Principle**
  - — effects of lower orders are more likely to be important than effects of higher orders; and
  - — effects of the same order are equally likely to be important
- **<u>Effect Sparsity</u> Principle (Pareto Principle)**
  - — the number of relatively important effects in a factorial experiment is small
- **<u>Effect Heredity</u> Principle**
  - — if an interaction is significant, then at least one of its parent factors is significant

½ fraction design: Remarks

- We used the defining equation

$$I = ABCD$$

in our example above.

- We could use the defining equation

$$I = -ABCD$$

equally well.

½ fraction design: Remarks

- We used the defining equation

$$I = ABCD$$

in our example above.

- We could also use

$$C = AB \qquad I = ABC$$

but the design would not be so good.

Fractional Factorial Design: Resolution of the design

Recall that the right-hand side of the defining equation, such as

$$I = ABCD \qquad \text{or} \qquad I = ABC$$

is called **the defining word.**

The length of the defining word is the number of the letters it consists of.

(Here the length is 4 or 3, respectively.)

The **resolution** of a fractional factorial design is

the minimum length of the defining words.

Fractional Factorial Design: Resolution of the design

The defining word is the right-hand side of the defining equation, such as

$$I = ABCD \qquad \text{or} \qquad I = ABC$$

The **resolution** of a fractional factorial design is

the minimum length of the defining words.

The resolution is usually denoted by Roman numerals (I, II, III, IV, V, etc.).

Here the resolution of the fractional design is

$$\text{IV} \qquad \text{or} \qquad \text{III}$$

respectively.

Fractional Factorial Design: Resolution of the design

In our example, the defining equation $I = ABCD$ yields one half fraction factorial design denoted by

$$2_{\mathrm{IV}}^{4-1}$$

The defining equation $I = ABC$ would yield one half fraction factorial design

$$2_{\mathrm{III}}^{4-1}$$

and we would get the following alias structure:

$$I = ABC \qquad A = BC$$

$$B = AC$$

$$C = AB$$

Fractional Factorial Design: Resolution of the design

Here the generating equation is $C = AB$

No.	I = ABC	**A** = BC	**B** = AC	**D** = ABCD	C = AB	AD = BCD	BD = ACD	CD = ABD
1	+	−	−	−	+	+	+	−
2	+	+	−	−	−	−	+	+
3	+	−	+	−	−	+	−	+
4	+	+	+	−	+	−	−	−
5	+	−	−	+	+	−	−	+
6	+	+	−	+	−	+	−	−
7	+	−	+	+	−	−	+	−
8	+	+	+	+	+	+	+	+

Here the interactions of the 2^{nd} order are confounded with

the main factors as well as with some interactions of the 3^{rd} order.

Fractional Factorial Design: Resolution of the design

Maximal Resolution Principle:

The higher the resolution is

the better the fractional factorial design is.

One quarter (¼) fraction design

¼ fraction design

- Consider k main two-level factors in the experiment.
- For example, consider $k = 5$ factors A, B, C, D, E in the experiment.
- We wish to have a one quarter fraction factorial design, i.e. $p = 2$,

  i.e. the factorial design 2^{5-2}
- Since $5 - 2 = 3$, we have to choose 3 main factors, such as A, B, C
- We then add the generating equations

$$D = AB \qquad \text{and} \qquad E = BC$$

- Notice that writing

$$D = ABC \qquad \text{and} \qquad E = BC \ldots ?$$

¼ fraction design

- Having the generating equations

$$D = AB \qquad \text{and} \qquad E = BC$$

- We get the defining equations

$$I = ABD \qquad \text{and} \qquad I = BCE$$

- RULE:

  The collection of the defining equations must be closed under multiplication.

- We thus multiply the equations together:

$$I = ACDE$$

¼ fraction design

- Now, the collection of the equations

$$I = ABD \qquad \text{and} \qquad I = BCE \qquad \text{and} \qquad I = ACDE$$

  is closed under multiplication.
- **The defining contrast subgroup is**

$$\{I,\ ABD,\ BCE,\ ACDE\}$$

- Notice that there are always $2^p = 2^2 = 4$ elements in the subgroup.
- The resolution is 3 = III.
- The resulting fractional design is

$$2^{5-2}_{\mathrm{III}}$$

¼ fraction design

Given the defining equations

$$I = ABD \quad \text{and} \quad I = BCE \quad \text{and} \quad I = ACDE$$

the resulting alias structure is:

$$I = ABD = BCE = ACDE$$

$$A = BD = ABCE = CDE$$

$$B = AD = CE = ABCDE$$

$$C = ABCD = BE = ADE$$

$$D = AB = BCDE = ACE$$

$$E = ABDE = BC = ACD$$

$$AC = BCD = ABE = DE$$

$$AE = BDE = ABC = CD$$

One eight (⅛) fraction design

⅛ fraction design

- Consider k main two-level factors in the experiment.
- For example, consider $k = 7$ factors A, B, C, D, E, F, G in the experiment.
- We wish to have a one eight fraction factorial design, i.e. $p = 3$,

  i.e. the factorial design 2^{7-3}
- Since $7 - 3 = 4$, we have to choose 4 main factors, such as A, B, C, D
- We then add the generating equations

  $$E = ABC \qquad \text{and} \qquad F = ABD \qquad \text{and} \qquad G = ACD$$

  for example.
- Other possibilities of the choice of the generating equations

⅛ fraction design

- Having the generating equations

$$E = ABC \qquad \text{and} \qquad F = ABD \qquad \text{and} \qquad G = ACD$$

- We get the defining equations

$$I = ABCE \qquad \text{and} \qquad I = ABDF \qquad \text{and} \qquad I = ACDG$$

- And the collection of defining equations closed under multiplication

$$I = ABCE \qquad \text{and} \qquad I = ABDF \qquad \text{and} \qquad I = ACDG$$

$$I = CDEF \qquad \text{and} \qquad I = BCFG \qquad \text{and} \qquad I = BDEG$$

$$I = AEFG$$

- The resulting fractional design is

For specialists in algebra: The general 2^{-p} fraction design

The general 2^{-p} fraction design

- Assume k factors.
- We wish to set up a 2^{-p} fractional design.
- Consider the index set

$$\mathcal{S} = \{\pm\}^k = \{+,-\}^k$$

- Notice that, in the set $\mathcal{S}$, there are 2^k elements of the form

$$S = (s_1, s_2, \ldots, s_k) \in \mathcal{S}$$

where $s_1, s_2, \ldots, s_k \in \{+,-\}$ are signs “+” or “−”.

The general 2^{-p} fraction design

- Consider now the "main" group

$$\mathcal{M} = \{\pm\}^{\mathcal{S}} = \{+,-\}^{\mathcal{S}}$$

- The group $\mathcal{M}$ consists of all the 2^k-component columns of signs "+" and "−".
- The operation of multiplication of the elements of $\mathcal{M}$ is defined componentwise, that is, given two elements

$$M = (m_S)_{S\in\mathcal{S}} \in \mathcal{M} \qquad \text{and} \qquad N = (n_S)_{S\in\mathcal{S}} \in \mathcal{M}$$

where $m_S \in \{+,-\}$ and $n_S \in \{+,-\}$ are signs "+" or "−" for $S \in \mathcal{S}$

- we have

$$M \times N = (m_S)_{S\in\mathcal{S}} \times (n_S)_{S\in\mathcal{S}} = (m_S \times n_S)_{S\in\mathcal{S}}$$

The general 2^{-p} fraction design

Given

$$M = (m_S)_{S \in \mathcal{S}} \in \mathcal{M} \qquad \text{and} \qquad N = (n_S)_{S \in \mathcal{S}} \in \mathcal{M}$$

we have

$$M \times N = (m_S)_{S \in \mathcal{S}} \times (n_S)_{S \in \mathcal{S}} = (m_S \times n_S)_{S \in \mathcal{S}}$$

by using the rules

$$+ \times + = + \qquad + \times - = - \qquad - \times + = - \qquad - \times - = +$$

Notice that the column

$$I = (+)_{S \in \mathcal{S}} \in \mathcal{M}$$

which corresponds to the intercept term and consists of 2^k plus ("+") signs,

is neutral with respect to the multiplication (i.e., it is the neutral element).

The general 2^{-p} fraction design

Notice that it holds

$$(M \times N) \times O = M \times (N \times O) \qquad \text{(associativity)}$$

$$M \times N = N \times M \qquad \text{(commutativity)}$$

$$M \times I = M = I \times M \qquad \text{(neutral element)}$$

$$M \times M = I \qquad \text{(self-inverse)}$$

for every

$$M = (m_S)_{S \in \mathcal{S}} \in \mathcal{M} \qquad \text{and} \qquad N = (n_S)_{S \in \mathcal{S}} \in \mathcal{M} \qquad \text{and} \qquad O = (o_S)_{S \in \mathcal{S}} \in \mathcal{M}$$

which implies that $\mathcal{M}$ is a commutative group (of a special form).

The general 2^{-p} fraction design

- Recall that an index $S \in \mathcal{S}$ is

$$S = (s_1, s_2, \dots, s_k)$$

where $s_1, s_2, \dots, s_k \in \{+, -\}$ are signs "+" or "–".

- Consider now the k columns

$$(s_1)_{S\in\mathcal{S}} \qquad (s_2)_{S\in\mathcal{S}} \qquad (s_3)_{S\in\mathcal{S}} \qquad \dots \qquad (s_k)_{S\in\mathcal{S}}$$

each consisting of 2^k components.

- These columns correspond to the k main factors:

$$\underbrace{(s_1)_{S\in\mathcal{S}}}_{A} \qquad \underbrace{(s_2)_{S\in\mathcal{S}}}_{B} \qquad \underbrace{(s_3)_{S\in\mathcal{S}}}_{C} \qquad \dots \qquad \underbrace{(s_k)_{S\in\mathcal{S}}}_{K}$$

The general 2^{-p} fraction design

- The k columns

$$(s_1)_{s\in\mathcal{S}} \qquad (s_2)_{s\in\mathcal{S}} \qquad (s_3)_{s\in\mathcal{S}} \qquad \dots \qquad (s_k)_{s\in\mathcal{S}}$$

  generate a subgroup

$$\mathcal{G} \quad \text{of} \quad \mathcal{M}$$

- The subgroup $\mathcal{G}$ contains the neutral element $I = (+)_{s\in\mathcal{S}}$, i.e. the intercept, as well as other columns corresponding to the interactions of the main factors, such as

$$\underbrace{(s_1)_{s\in\mathcal{S}} \times (s_2)_{s\in\mathcal{S}}}_{\text{AB}} \qquad \underbrace{(s_1)_{s\in\mathcal{S}} \times (s_3)_{s\in\mathcal{S}}}_{\text{AC}} \qquad \underbrace{(s_1)_{s\in\mathcal{S}} \times (s_2)_{s\in\mathcal{S}} \times (s_3)_{s\in\mathcal{S}}}_{\text{ABC}} \qquad \text{etc.}$$

The general 2^{-p} fraction design

- Notice that the subgroup $\mathcal{G}$ corresponds to the full factorial design of the experiment.

- Considering the k main columns

$$(s_1)_{s\in\mathcal{S}} \qquad (s_2)_{s\in\mathcal{S}} \qquad (s_3)_{s\in\mathcal{S}} \qquad \dots \qquad (s_k)_{s\in\mathcal{S}}$$

the full factorial experiment will be carried out for each combination

$$s_1, s_2, s_3, \dots, s_k$$

of the levels of the main factors $\underbrace{\text{A, B, C, …, K}}_{k \text{ factors}}$ for each $s = (s_1, s_2, s_3, \dots, s_k) \in \mathcal{S}$.

The general 2^{-p} fraction design

- There is also the column

$$-I = (-)_{s \in \mathcal{S}} \in \mathcal{M}$$

which consists of 2^k minus ("–") signs, in the "main" group $\mathcal{M}$.

- Having the subgroup $\mathcal{G} \triangleleft \mathcal{M}$ of 2^k elements, extend the subgroup $\mathcal{G}$ to the subgroup $\tilde{\mathcal{G}} \triangleleft \mathcal{M}$ defined as follows:

$$\tilde{\mathcal{G}} = \mathcal{G} \cup \{-I \times G : G \in \mathcal{G}\}$$

- Notice that there are $2 \times 2^k = 2^{k+1}$ elements in the group $\tilde{\mathcal{G}}$.

The general 2^{-p} fraction design

- Now, let a fraction index $p \geq 0$ be given.
- Choose any subgroup

$$\tilde{\mathcal{H}} \quad \text{of} \quad \tilde{\mathcal{G}}$$

such that

there are 2^p elements in the subgroup $\tilde{\mathcal{H}}$

and

$$-I \notin \tilde{\mathcal{H}}$$

- The subgroup $\tilde{\mathcal{H}}$ is the **defining equations subgroup.**

The general 2^{-p} fraction design

- Now, let a fraction index $p \geq 0$ be given.
- Let the defining equations subgroup $\tilde{\mathcal{H}} \lhd \tilde{\mathcal{G}}$

  be such that $-I \notin \tilde{\mathcal{H}}$ and $|\tilde{\mathcal{H}}| = 2^k$.

- If $p = 0$, then the subgroup $\tilde{\mathcal{H}}$ contains the element I only $(\mathcal{H} = \{I\})$.
- If $p = 1$, then there are $2^1 = 2$ elements in the group $\tilde{\mathcal{H}}$, i.e. the intercept I

  and yet another element.
- If $p = 2$, then there are $2^2 = 4$ elements in the group $\tilde{\mathcal{H}}$, i.e. the intercept I

  and yet three other elements.

The general 2^{-p} fraction design

- Now, let a fraction index $p \geq 0$ be given.
- Given the defining equations subgroup $\tilde{\mathcal{H}} \lhd \tilde{\mathcal{G}}$ such that

$$-I \notin \tilde{\mathcal{H}} \qquad \text{and} \qquad \left|\tilde{\mathcal{H}}\right| = 2^k$$

- introduce the **defining contrast subgroup** as follows:
- For every $\tilde{H} \in \tilde{\mathcal{H}}$,
  - if $\tilde{H} \in \mathcal{G}$, then let $\tilde{H} \in \mathcal{H}$
  - if $\tilde{H} \notin \mathcal{G}$, then let $-I \times \tilde{H} \in \mathcal{H}$

The general 2^{-p} fraction design

- The defining contrast subgroup $\mathcal{H}$, which we have chosen, induces the **alias structure.**
- The alias structure is set up according to the factor group

$$\mathcal{G}/\mathcal{H}$$

- Notice that there are

$$|\mathcal{G}|/|\mathcal{H}| = 2^n/2^p = 2^{n-p}$$

elements in the factor group $\mathcal{G}/\mathcal{H}$.

The general 2^{-p} fraction design: Example

Consider 4 main factors A, B, C, D.

Choosing the defining equations subgroup

$$\tilde{\mathcal{H}} = \{I, ABCD\} \quad \text{or} \quad \tilde{\mathcal{H}} = \{I, -ABCD\}$$

we obtain the defining contrast subgroup

$$\mathcal{H} = \{I, ABCD\}$$

and the alias structure

$$\mathcal{G}/_{\mathcal{H}} = \left\{ \begin{matrix} \{I, ABCD\}, \\ \{A, BCD\}, \ \{B, ACD\}, \ \{C, ABD\}, \ \{D, ABC\}, \\ \{AB, CD\}, \ \{AC, BD\}, \ \{AD, BC\} \end{matrix} \right\}$$

The general 2^{-p} fraction design: Example

Consider 4 main factors A, B, C, D.

Choosing the defining equations subgroup

$$\tilde{\mathcal{H}} = \{I, ABC\} \quad \text{or} \quad \tilde{\mathcal{H}} = \{I, -ABC\}$$

we obtain the defining contrast subgroup

$$\mathcal{H} = \{I, ABC\}$$

and the alias structure

$$\mathcal{G}/\mathcal{H} = \left\{ \begin{array}{c} \{I, ABC\}, \\ \{A, BC\}, \{B, AC\}, \{C, AB\}, \{D\}, \\ \{AD, BCD\}, \{BD, ACD\}, \{CD, ABD\} \end{array} \right\}$$

The general 2^{-p} fraction design: Example

Consider 5 main factors A, B, C, D, E.

Choosing the defining equations subgroup

$$\tilde{\mathcal{H}} = \{I, ABD, BCE, ACDE\}$$

or

$$\tilde{\mathcal{H}} = \{I, -ABD, -BCE, ACDE\}$$

or

$$\tilde{\mathcal{H}} = \{I, -ABD, BCE, -ACDE\}$$

or

$$\tilde{\mathcal{H}} = \{I, ABD, -BCE, -ACDE\}$$

we obtain the defining contrast subgroup

$$\mathcal{H} = \{I, ABD, BCE, ACDE\}$$

The general 2^{-p} fraction design: Example

Consider 5 main factors A, B, C, D, E.

Given the defining contrast subgroup

$$\mathcal{H} = \{I, ABD, BCE, ACDE\}$$

we obtain the alias structure

$$\mathcal{G}/_{\mathcal{H}} = \left\{ \begin{array}{c} \{I, ABD, BCE, ACDE\}, \\ \{A, BD, ABCE, CDE\},\ \{B, AD, CE, ABCDE\},\ \{C, ABCD, BE, ADE\}, \\ \{D, AB, BCDE, ACE\},\ \{E, ABDE, BCDE, ACD\}, \\ \{AC, BCD, ABE, DE\},\ \{AE, BDE, ABC, CD\} \end{array} \right\}$$

The general 2^{-p} fraction design: Example

Consider 7 main factors A, B, C, D, E, F, G.

Choosing the defining equations subgroup

$$\tilde{\mathcal{H}} = \{I, ABCE, ABDF, ACDG, CDEF, BCFG, BDEG, AEFG\}$$

or

$$\tilde{\mathcal{H}} = \{I, ABCE, ABDF, -ACDG, CDEF, -BCFG, -BDEG, -AEFG\}$$

or

$$\tilde{\mathcal{H}} = \{I, ABCE, -ABDF, ACDG, -CDEF, -BCFG, BDEG, -AEFG\}$$

or

$$\tilde{\mathcal{H}} = \{I, ABCE, -ABDF, -ACDG, -CDEF, BCFG, -BDEG, AEFG\}$$

or

…

The general 2^{-p} fraction design: Example

Consider 7 main factors A, B, C, D, E, F, G.

Choosing the defining equations subgroup

...

or

$$\tilde{\mathcal{H}} = \{I, -ABCE, ABDF, ACDG, -CDEF, BCFG, -BDEG, -AEFG\}$$

or

$$\tilde{\mathcal{H}} = \{I, -ABCE, ABDF, -ACDG, -CDEF, -BCFG, BDEG, AEFG\}$$

or

$$\tilde{\mathcal{H}} = \{I, -ABCE, -ABDF, ACDG, CDEF, -BCFG, -BDEG, AEFG\}$$

or

$$\tilde{\mathcal{H}} = \{I, -ABCE, -ABDF, -ACDG, CDEF, BCFG, BDEG, -AEFG\}$$

The general 2^{-p} fraction design: Example

Consider 7 main factors A, B, C, D, E, F, G.

Choosing the defining equations subgroup as above,

we obtain the defining contrast subgroup

$$\mathcal{H} = \{I, ABCE, ABDF, ACDG, CDEF, BCFG, BDEG, AEFG\}$$

and the alias structure

(see the next slide)

The general 2^{-p} fraction design: Example

$$\mathcal{G}/\mathcal{H} = \left\{\begin{array}{l}
\{I,\ ABCE, ABDF, ACDG, CDEF, BCFG, BDEG, AEFG\}, \\
\{A, BCE, BDF, CDG, ACDEF, ABCFG, ABDEG, EFG\}, \\
\{B, ACE, ADF, ABCDG, BCDEF, CFG, DEG, ABEFG\}, \\
\{C, ABE, ABCDF, ADG, DEF, BFG, BCDEG, ACEFG\}, \\
\{D, ABCDE, ABF, ACG, CEF, BCDEF, BEG, ADEFG\}, \\
\{E, ABC, ABDEF, ACDEG, CDF, BCEFG, BDG, AFG\}, \\
\{F, ABCEF, ABD, ACDFG, CDE, BCG, BDEFG, AEG\}, \\
\{G, ABCEG, ABDFG, ACD, CDEFG, BCF, BDE, AEF\}, \\
\{AB, CE, DF, BCDG, ABCDEF, ACFG, ADEG, BEFG\}, \\
\{AC, BE, BCDF, DG, ADEF, ABFG, ABCDEG, CEFG\}, \\
\{AD, BCDE, BF, CG, ACEF, ABCDFG, ABEG, DEFG\}, \\
\{AE, BC, BDEF, CDEG, ACDF, ABCEFG, ABDG, FG\}, \\
\{AF, BCEF, BD, CDFG, ACDE, ABCG, ABDEFG, EG\}, \\
\{AG, BCEG, BDFG, CD, ACDEFG, ABCF, ABDE, EF\}, \\
\{BG, ACEG, ADFG, ABCD, BCDEFG, CF, DE, ABEF\}, \\
\{ABCDEFG, DFG, CEG, BEF, ABG, ADE, ACF, BCD\}
\end{array}\right\}$$

The general 2^{-p} fraction design

Since the subgroup $\mathcal{G} \lhd \mathcal{M}$ is generated by the columns

$$\underbrace{(s_1)_{s\in\mathcal{S}}}_{A} \qquad \underbrace{(s_2)_{s\in\mathcal{S}}}_{B} \qquad \underbrace{(s_3)_{s\in\mathcal{S}}}_{C} \qquad \cdots \qquad \underbrace{(s_k)_{s\in\mathcal{S}}}_{K}$$

and $\mathcal{H}$ is a subgroup of $\mathcal{G}$, i.e. $\mathcal{H} \lhd \mathcal{G} \lhd \mathcal{M}$, it follows that each element of $\mathcal{H}$ can be generated by these columns. That is,

— for every element $H \in \mathcal{H}$, we have

$$H = \prod_{j\in J_H} (s_j)_{s\in\mathcal{S}} \qquad \text{for some} \quad J_H \subseteq \{1, 2, \dots, k\}$$

The empty set $J_I = \emptyset$ yields the intercept I.

The general 2^{-p} fraction design

The **resolution** of the defining contrast subgroup $\mathcal{H}$ is the minimum length of a word in it (not counting the element "I").

The resolution is written as a Roman number (I, II, III, IV, V, ...).

For example, the resolution of the defining contrast subgroup

$$\mathcal{H} = \{I, ABD, BCE, ACDE\}$$

is 3 = III.

For example, the resolution of the defining contrast subgroup

$$\mathcal{H} = \{I, ABCE, ABDF, ACDG, CDEF, BCFG, BDEG, AEFG\}$$

The general 2^{-p} fraction design

The **resolution** of the defining contrast subgroup $\mathcal{H}$ is the minimum length of a word in it (not counting the element "I"). The resolution is written as a Roman number (I, II, III, IV, V, …).

For example, the resolution of both the defining contrast subgroups

$$\mathcal{H} = \{I, ABCF, ABDEG, CDEFG\}$$

$$\mathcal{H} = \{I, ABCF, ADEG, BCDEFG\}$$

is 4 = IV.

The general 2^{-p} fraction design

The **resolution** of the defining contrast subgroup $\mathcal{H}$ is the minimum length of a word in it (not counting the element "I").

The **aberration** of the defining contrast subgroup $\mathcal{H}$ is the count of the distinct minimum length words in the subgroup.

For example, the resolution of the defining contrast subgroup

$$\mathcal{H} = \{I, ABD, BCE, ACDE\}$$

is 3 = III and its aberration is 2.

The general 2^{-p} fraction design

The **resolution** of the defining contrast subgroup $\mathcal{H}$ is the minimum length of a word in it (not counting the element "I").

The **aberration** of the defining contrast subgroup $\mathcal{H}$ is the count of the distinct minimum length words in the subgroup.

For example, the resolution of the defining contrast subgroup

$$\mathcal{H} = \{I, ABCE, ABDF, ACDG, CDEF, BCFG, BDEG, AEFG\}$$

is 4 = IV and its aberration is 7.

The general 2^{-p} fraction design

- The purpose is to choose the defining equation subgroup $\tilde{\mathcal{H}}$ so that **<u>the resolution</u>** of the defining contrast subgroup $\mathcal{H}$ **<u>is maximized</u>**.

- If there are several defining equation subgroups $\tilde{\mathcal{H}}$ such that the corresponding defining contrast subgroups have the same maximum resolution, then the purpose is to choose the defining equation subgroup $\tilde{\mathcal{H}}$ so that **<u>the aberration</u>** of the defining contrast subgroup $\mathcal{H}$ **<u>is minimized</u>**.

The general 2^{-p} fraction design

For example, if the defining contrast subgroup is

$$\mathcal{H} = \{I, ABCF, ABDEG, CDEFG\}$$

the resolution is 4 = IV and the aberration is 1.

If the defining contrast subgroup is

$$\mathcal{H} = \{I, ABCF, ADEG, BCDEFG\}$$

the resolution is also 4 = IV, but the aberration is 2.

The former subgroup is better because its aberration is less than that of the latter.

The general 2^{-p} fraction design

- **It remains to set up the fractional factorial design of the experiment.**
- Our purpose is to set up the 2^{-p} fraction design, denoted by

$$2^{k-p}$$

The procedure is as follows

(see the following slides).

The general 2^{-p} fraction design

- Write every word $\tilde{H} \in \tilde{\mathcal{H}}$ (except the element "I")

  of the defining equation subgroup $\tilde{\mathcal{H}}$ in the form

$$\tilde{H} = \prod_{j \in J_{\tilde{H}}} (s_j)_{s \in \mathcal{S}} \quad \text{or} \quad \tilde{H} = -\prod_{j \in J_{\tilde{H}}} (s_j)_{s \in \mathcal{S}} \quad \text{for exactly one} \quad J_H \subseteq \{1, 2, \dots, k\}$$

- Write the corresponding **defining equation:**

$$+ = \prod_{j \in J_{\tilde{H}}} s_j \quad \text{or} \quad + = -\prod_{j \in J_{\tilde{H}}} s_j \qquad (*)$$

respectively, where "s_j" are symbols (variables).

The general 2^{-p} fraction design

For example, if the defining equation subgroup is

$$\tilde{\mathcal{H}} = \{I, -ABD, -BCE, ACDE\}$$

then the collection of the defining equations is

$$I = -ABD \qquad I = -BCE \qquad I = ACDE$$

The general 2^{-p} fraction design

The group $\mathcal{G}$ is generated by the columns

$$\underbrace{(s_1)_{s\in\mathcal{S}}}_{\text{A}} \qquad \underbrace{(s_2)_{s\in\mathcal{S}}}_{\text{B}} \qquad \underbrace{(s_3)_{s\in\mathcal{S}}}_{\text{C}} \qquad \cdots \qquad \underbrace{(s_k)_{s\in\mathcal{S}}}_{\text{K}}$$

corresponding to the k main factors A, B, C, …, K,
and there are 2^k elements in the group $\mathcal{G}$.

- The group $\mathcal{G}$ can be seen a $2^k \times 2^k$ table thus.
- A row with the index $S = (s_1, s_2, \dots, s_k) \in \mathcal{S}$ is kept – and the experiment is carried out for this combination of the levels of the main factors – if and only if all the defining equations $(*)$ are fulfilled.

The general 2^{-p} fraction design

- Finally, the effects of the factors and their interactions are confounded according to the defining equations subgroup $\tilde{\mathcal{H}}$.

To given an example, consider 5 main factors

A, B, C, D, E

and let the defining equations subgroup be

$$\tilde{\mathcal{H}} = \{I, ABD, -BCE, -ACDE\}$$

The resolution is 3 = III and the aberration is 2.

The general 2^{-p} fraction design: Example

Full factorial design 2^5: the group $\mathcal{G}$

No.	I	A	B	C	D	E	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE	ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE	ABCD	ABCE	ABDE	ACDE	BCDE	ABCDE
1	+	−	−	−	−	−	+	+	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	+	+	+	+	+	−
2	+	+	−	−	−	−	−	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−	+	+
3	+	−	+	−	−	−	−	+	+	+	−	−	−	+	+	+	+	+	+	−	−	−	+	+	+	−	−	−	−	+	−	+
4	+	+	+	−	−	−	+	−	−	−	−	−	−	+	+	+	−	−	−	+	+	+	+	+	+	−	+	+	+	−	−	−
5	+	−	−	+	−	−	+	−	+	+	−	+	+	−	−	+	+	−	−	+	+	−	+	+	−	+	−	−	+	−	−	+
6	+	+	−	+	−	−	−	+	−	−	−	+	+	−	−	+	−	+	+	−	−	+	+	+	−	+	+	+	−	+	−	−
7	+	−	+	+	−	−	−	−	+	+	+	−	−	−	−	+	−	+	+	+	+	−	−	−	+	+	+	+	−	−	+	−
8	+	+	+	+	−	−	+	+	−	−	+	−	−	−	−	+	+	−	−	−	−	+	−	−	+	+	−	−	+	+	+	+
9	+	−	−	−	+	−	+	+	−	+	+	−	+	−	+	−	−	+	−	+	−	+	+	−	+	+	−	+	−	−	−	+
10	+	+	−	−	+	−	−	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	+	+	−	+	+	−	−
11	+	−	+	−	+	−	−	+	−	+	−	+	−	−	+	−	+	−	+	+	−	+	−	+	−	+	+	−	+	−	+	−
12	+	+	+	−	+	−	+	−	+	−	−	+	−	−	+	−	−	+	−	−	+	−	−	+	−	+	−	+	−	+	+	+
13	+	−	−	+	+	−	+	−	−	+	−	−	+	+	−	−	+	+	−	−	+	+	−	+	+	−	+	−	−	+	+	−
14	+	+	−	+	+	−	−	+	+	−	−	−	+	+	−	−	−	−	+	+	−	−	−	+	+	−	−	+	+	−	+	+
15	+	−	+	+	+	−	−	−	−	+	+	+	−	+	−	−	−	−	+	−	+	+	+	−	−	−	−	+	+	+	−	+
16	+	+	+	+	+	−	+	+	+	−	+	+	−	+	−	−	+	+	−	+	−	−	+	−	−	−	+	−	−	−	−	−
17	+	−	−	−	−	+	+	+	+	−	+	+	−	+	−	−	−	−	+	−	+	+	−	+	+	+	+	−	−	−	−	+
18	+	+	−	−	−	+	−	−	−	+	+	+	−	+	−	−	+	+	−	+	−	−	−	+	+	+	−	+	+	+	−	−
19	+	−	+	−	−	+	−	+	+	−	−	−	+	+	−	−	+	+	−	−	+	+	+	−	−	+	−	+	+	−	+	−
20	+	+	+	−	−	+	+	−	−	+	−	−	+	+	−	−	−	−	+	+	−	−	+	−	−	+	+	−	−	+	+	+
21	+	−	−	+	−	+	+	−	+	−	−	+	−	−	+	−	+	−	+	+	−	+	+	−	+	−	−	+	−	+	+	−
22	+	+	−	+	−	+	−	+	−	+	−	+	−	−	+	−	−	+	−	−	+	−	+	−	+	−	+	−	+	−	+	+
23	+	−	+	+	−	+	−	−	+	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+	−	−	+	−	+	+	−	+
24	+	+	+	+	−	+	+	+	−	+	+	−	+	−	+	−	+	−	+	−	+	−	−	+	−	−	−	+	−	−	−	−
25	+	−	−	−	+	+	+	+	−	−	+	−	−	−	−	+	−	+	+	+	+	−	+	+	−	−	−	−	+	+	+	−
26	+	+	−	−	+	+	−	−	+	+	+	−	−	−	−	+	+	−	−	−	−	+	+	+	−	−	+	+	−	−	+	+
27	+	−	+	−	+	+	−	+	−	−	−	+	+	−	−	+	+	−	−	+	+	−	−	−	+	−	+	+	−	+	−	+
28	+	+	+	−	+	+	+	−	+	+	−	+	+	−	−	+	−	+	+	−	−	+	−	−	+	−	−	−	+	−	−	−
29	+	−	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	−	+
30	+	+	−	+	+	+	−	+	+	+	−	−	−	+	+	+	−	−	−	+	+	+	−	−	−	+	−	−	−	+	−	−
31	+	−	+	+	+	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	−	−	−	+	−
32	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

The general 2^{-p} fraction design: Example

Given the defining equations subgroup

$$\tilde{\mathcal{H}} = \{I, ABD, -BCE, -ACDE\}$$

the defining contrast subgroup is

$$\mathcal{H} = \{I, ABD, BCE, ACDE\}$$

and the alias structure is

$$\mathcal{G}/\mathcal{H} = \left\{\begin{matrix} \{I, ABD, BCE, ACDE\}, \\ \{A, BD, ABCE, CDE\}, \ \{B, AD, CE, ABCDE\}, \ \{C, ABCD, BE, ADE\}, \\ \{D, AB, BCDE, ACE\}, \ \{E, ABDE, BC, ACD\}, \\ \{AC, BCD, ABE, DE\}, \ \{AE, BDE, ABC, CD\} \end{matrix}\right\}$$

The general 2^{-p} fraction design: Example

Fractional factorial design 2_{III}^{5-2}

The defining equations:

$$I = ABD \qquad I = -BCE \qquad I = -ACDE$$

No.	I + ABD − BCE − ACDE	**A** + BD − ABCE − CDE	**B** + AD − CE − ABCDE	**C** + ABCD − BE − ADE	**D** + AB − BCDE − ACE	**E** + ABDE − BC − ACD	AC + BCD − ABE − DE	AE + BDE − ABC − CD
1 (□ 2)	+	+	−	−	−	−	−	−
2 (□ 7)	+	−	+	+	−	−	−	+
3 (□ 9)	+	−	−	−	+	−	+	+
4 (16)	+	+	+	+	+	−	+	−
5 (19)	+	−	+	−	−	+	+	−
6 (22)	+	+	−	+	−	+	+	+
7 (28)	+	+	+	−	+	+	−	+
8 (29)	+	−	−	+	+	+	−	−