Statistical Methods for Economists

Lecture 9

Full Factorial Design of Experiments

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Outline of the lecture

- Motivation & Introduction
- Full Factorial Experiments
- Graphical assessment of factor / interaction significance
- Graphs of interaction effects

Full Factorial Experiments: Motivation

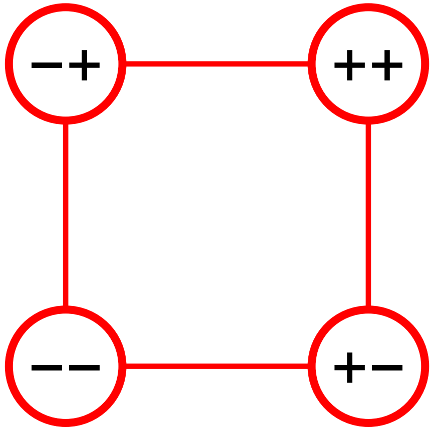
- We consider an observed variable Y, such as the quality of a product.
- We also consider several factors A, B, C, D, …, such as the conditions during the production.
- And we conjecture that the levels of the factors (and the interactions between the factors) influence the response variable Y, which we wish to maximize, say.
- The purpose is to identify the significant factors that influence the response variable Y most, and to find their optimal levels.
- We study the effect of the factors A, B, C, D, … on the response variable Y

Full Factorial Experiments: Motivation & Introduction

- We study the effect of the factors A, B, C, D, … on the response variable Y by means of experiment.
- We consider only a few (a finite number) of distinct values of each of the factors.
- We usually consider only two distinct values of each of the factors.
- The first value is "low" and is denoted by "−".
- The second value is "high" and is denoted by "+".
- If n factors are considered, then there are 2^n various combinations of the levels of the factors.

Full Factorial Experiments: Motivation & Introduction

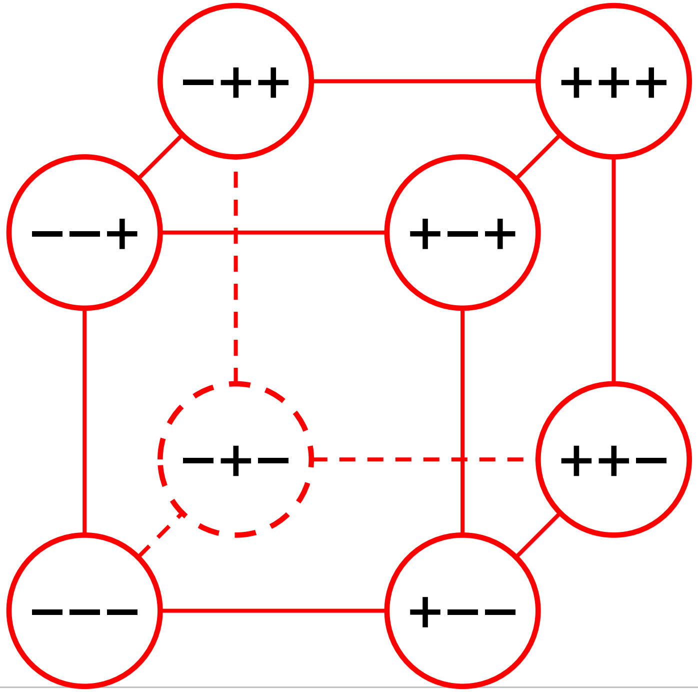
- If we consider $n = 2$ factors A and B, then the arrangement of the factor levels is called a 2×2 or 2^2 factorial design.
- There are $2 \times 2 = 4$ factorial points, which constitute a square:



Factor	
A	B
−	−
+	−
−	+
+	+

Full Factorial Experiments: Motivation & Introduction

- If we consider $n = 3$ factors A, B, C, then the arrangement of the factor levels is called a 2×2×2 or 2^3 factorial design.
- There are 2×2×2 = 8 factorial points, which constitute a cube:



Factor		
A	B	C
−	−	−
+	−	−
−	+	−
+	+	−
−	−	+
+	−	+
−	+	+
+	+	+

Full Factorial Experiments: Motivation & Introduction

- The theory of the Full Factorial Experiments is based upon the theory of Multiple Linear Regression (again).
- The theory and all the calculations are simplified significantly if we assume that the number of the experiments carried out in each case is the same.

- In other words, considering n factors and assuming that the experiment is repeated K-times for each combination of the factors, we have

$$2^n \times K$$

observations of the experiment in total.

Full Factorial Experiments: Motivation & Introduction

- In other words, considering n factors and assuming that the experiment is repeated K-times for each combination of the factors, we have

$$2^n \times K$$

numerical outcomes of the mutually independent random variables

$$Y_{sk} \sim \mathcal{N}(\mu_s, \sigma^2) \qquad \text{for} \quad s \in \mathcal{S} \quad \text{and for} \quad k = 1, 2, \dots, K$$

where

$$\mathcal{S} = \{\pm\}^n = \{+, -\}^n$$

is the index set of all the 2^n possible combinations of the levels of the factors.

Full Factorial Experiments: Motivation & Introduction

We thus have a sample

$$y_{s1}, y_{s2}, \dots, y_{sK} \qquad \text{for} \quad s \in \mathcal{S}$$

of the observations of the mutually independent random variables

$$Y_{s1}, Y_{s2}, \dots, Y_{sK} \sim \mathcal{N}(\mu_s, \sigma^2) \qquad \text{for} \quad s \in \mathcal{S}$$

where $\mathcal{S}$ is the index set of all the 2^n combinations of the levels of the factors.

(Here, the expected values μ_s and the variance σ^2 are unknown.

The variance σ^2 is assumed to be the same – homoskedasticity.)

Full Factorial Experiments: Example

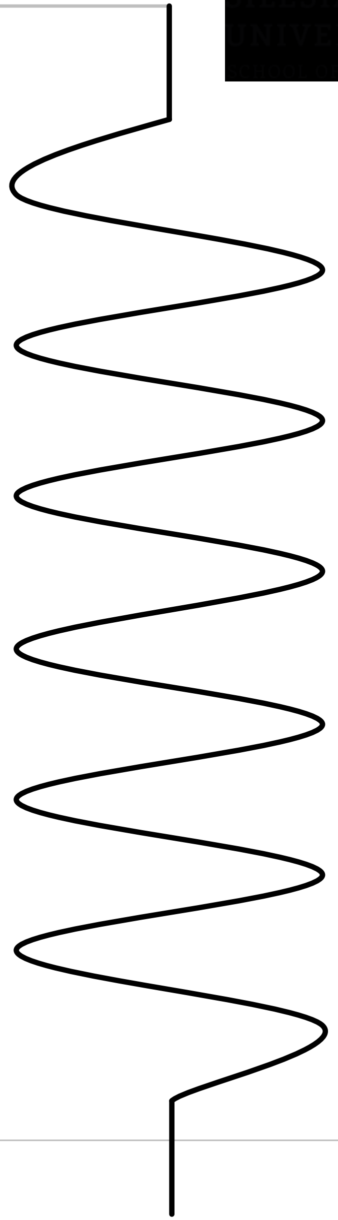
Consider a *spring*.

The observed variable Y is the lifespan of the spring,

i.e. the number of pressures of the spring until it cracks.

We consider three factors:

- Factor L = the length of the spring
- Factor G = the thickness of the wire of the spring
- Factor T = the material of (the wire of) the spring



Full Factorial Experiments: Example

We consider the following levels of the three factors:

- Factor L = the length of the spring

  "−" or "−1" = 10 cm

  "+" or "+1" = 15 cm

- Factor G = the thickness of the wire of the spring

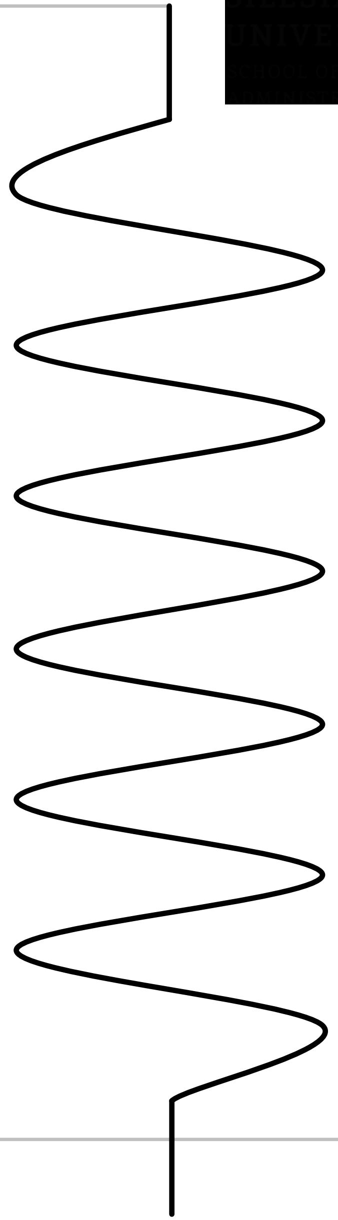
  "−" or "−1" = 5 mm

  "+" or "+1" = 7 mm

- Factor T = the material of (the wire of) the spring

  "−" or "−1" = Material / alloy "A"

  "+" or "+1" = Material / alloy "B"



Full Factorial Experiments: Example

The experiment was carried out 2× for each combination of the factors:

Factor			Observed value	
L	G	T	$Y_{\pm\pm\pm 1}$	$Y_{\pm\pm\pm 2}$
−	−	−	77	81
+	−	−	98	96
−	+	−	76	74
+	+	−	90	94
−	−	+	63	65
+	−	+	82	86
−	+	+	72	74
+	+	+	92	88

Full Factorial Experiments: Example

Now, the goal is to identify the factors, including their interactions, with significant effect on the lifespan of the spring, i.e. the observed values of the variable Y.

We shall use the theory of Multiple Linear Regression to this end.

We assume the model with the intercept term:

$$Y = 1\beta_0 + x_{\mathrm{L}}\beta_{\mathrm{L}} + x_{\mathrm{G}}\beta_{\mathrm{G}} + x_{\mathrm{T}}\beta_{\mathrm{T}} + x_{\mathrm{LG}}\beta_{\mathrm{LG}} + x_{\mathrm{LT}}\beta_{\mathrm{LT}} + x_{\mathrm{GT}}\beta_{\mathrm{GT}} + x_{\mathrm{LGT}}\beta_{\mathrm{LGT}}$$

Full Factorial Experiments: Example

We assume the model with the intercept term:

$$Y = 1\beta_0 + x_{\mathrm{L}}\beta_{\mathrm{L}} + x_{\mathrm{G}}\beta_{\mathrm{G}} + x_{\mathrm{T}}\beta_{\mathrm{T}} + x_{\mathrm{LG}}\beta_{\mathrm{LG}} + x_{\mathrm{LT}}\beta_{\mathrm{LT}} + x_{\mathrm{GT}}\beta_{\mathrm{GT}} + x_{\mathrm{LGT}}\beta_{\mathrm{LGT}}$$

The effect of the factor or the interaction **is significant** if and only if the respective coefficient β is non-zero.

→ We shall use the *t*-test for the respective parameter β (see Theorem 6) to this end.

Full Factorial Experiments: Example

We set up the matrix X as the first step:

"−" = "−1" & "+" = "+1"

Interaction = the product of the Factors.

Intercept	Factor			Interaction			
0	L	G	T	LG	LT	GT	LGT
1	−	−	−	+	+	+	−
1	+	−	−	−	−	+	+
1	−	+	−	−	+	−	+
1	+	+	−	+	−	−	−
1	−	−	+	+	−	−	+
1	+	−	+	−	+	−	−
1	−	+	+	−	−	+	−
1	+	+	+	+	+	+	+

This is a $2^n \times 2^n$ matrix in general.

Full Factorial Experiments: Example

Since we carried out each experiment 2× for each combination of the factors, we double each row of the table:

Interaction = the product of the Factors.

X: / **y:**

Intercept	Factor			Interaction				Observed
0	L	G	T	LG	LT	GT	LGT	$Y_{\pm\pm\pm k}$
1	−1	−1	−1	+1	+1	+1	−1	77
1	−1	−1	−1	+1	+1	+1	−1	81
1	+1	−1	−1	−1	−1	+1	+1	98
1	+1	−1	−1	−1	−1	+1	+1	96
1	−1	+1	−1	−1	+1	−1	+1	76
1	−1	+1	−1	−1	+1	−1	+1	74
1	+1	+1	−1	+1	−1	−1	−1	90
1	+1	+1	−1	+1	−1	−1	−1	94
1	−1	−1	+1	+1	−1	−1	+1	63
1	−1	−1	+1	+1	−1	−1	+1	65
1	+1	−1	+1	−1	+1	−1	−1	82
1	+1	−1	+1	−1	+1	−1	−1	86
1	−1	+1	+1	−1	−1	+1	−1	72
1	−1	+1	+1	−1	−1	+1	−1	74
1	+1	+1	+1	+1	+1	+1	+1	92
1	+1	+1	+1	+1	+1	+1	+1	88

This is a $(2^n \times K) \times (2^n)$ matrix in general.

Full Factorial Experiments

We shall perform the *t*-test for each individual parameter β.

Recall the Corollary of Theorem 6:

If $\beta_j = 0$, then

$$T = \frac{b_j}{\sqrt{s^2}\sqrt{c_{jj}}} \sim t_{N-\mathrm{rank}(X)}$$

where

- b_j is the estimate of the parameter β_j
- s^2 is the residual variance = mean square error
- c_{jj} the j-th element on the diagonal of the matrix $C = \left(X^{\mathrm{T}}X\right)^{-1}$
- N is the total number of observations, we have $N = 2^n \times K$ here

Full Factorial Experiments

Having recalled the Corollary of Theorem 6:

$$\text{if} \quad \beta_j = 0 \qquad \text{then} \quad T = \frac{b_j}{\sqrt{s^2}\sqrt{c_{jj}}} \sim t_{N-\operatorname{rank}(\boldsymbol{X})}$$

— the only task is to calculate everything in this special case, i.e. for this matrix $\boldsymbol{X}$.

Calculating, we obtain that:

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} = \begin{pmatrix} N & & & \\ & N & & \\ & & \ddots & \\ & & & N \end{pmatrix}$$

i.e. the $2^n \times 2^n$ diagonal matrix with the number $N = 2^n \times K$ on the diagonal.

Full Factorial Experiments: Example

In our example, we have $n = 3$ factors (L, G, T),

and each experiment is replicated $K = 2$ times. We thus have

$$N = 2^n \times K = 16$$

Moreover, the result $X^{\mathrm{T}}X$ is a $2^n \times 2^n$, i.e. 8×8, matrix

with the number 16 on its diagonal:

$$X^{\mathrm{T}}X = \begin{pmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}$$

Full Factorial Experiments

Calculating, we obtain that:

$$C = (X^{\mathrm{T}}X)^{-1} = \begin{pmatrix} 1/N & & & \\ & 1/N & & \\ & & \ddots & \\ & & & 1/N \end{pmatrix}$$

i.e. the $2^n \times 2^n$ diagonal matrix with the number $c_{jj} = 1/N = 1/(2^n \times K)$ on the diagonal.

Moreover, the estimates are

$$b = CX^{\mathrm{T}}y = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y = \frac{1}{N}X^{\mathrm{T}}y$$

Full Factorial Experiments: Example

In our example, we

$$C = (X^T X)^{-1} = \begin{pmatrix} 1/16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/16 \end{pmatrix}$$

And the estimates are:

— the intercept term = the sample mean of the observed values:

$$b_0 = \frac{1}{N}(X_0)^T y = \frac{1}{16}(+77+98+76+90+63+82+72+92+81+96+74+94+65+86+74+88) =$$

$$= 1308/16 = 81.75$$

Full Factorial Experiments: Example

The estimates of the effects of the factors are:

$$b_L = \frac{1}{N}(X_L)^T y = \frac{1}{16}(-77 - 98 + 76 + 90 - 63 - 82 + 72 + 92 - 81 - 96 + 74 + 94 - 65 - 86 + 74 + 88) =$$

$$= 144/16 = 9$$

$$b_G = \frac{1}{N}(X_G)^T y = \frac{1}{16}(-77 - 98 - 76 - 90 + 63 + 82 + 72 + 92 - 81 - 96 - 74 - 94 + 65 + 86 + 74 + 88) =$$

$$= 12/16 = 0.75$$

$$b_T = \frac{1}{N}(X_T)^T y = \frac{1}{16}(-77 - 98 - 76 - 90 - 63 - 82 - 72 - 92 + 81 + 96 + 74 + 94 + 65 + 86 + 74 + 88) =$$

$$= -64/16 = -4$$

Full Factorial Experiments: Example

The estimates of the effects of the interactions between / among the factors are:

$$b_{LG} = \frac{1}{N}(X_{LG})^T y = \frac{1}{16}(+77 + 98 - 76 - 90 - 63 - 82 + 72 + 92 + 81 + 96 - 74 - 94 - 65 - 86 + 74 + 88) =$$

$$= -8/16 = \quad -0.5$$

$$b_{LT} = \frac{1}{N}(X_{LT})^T y = \frac{1}{16}(+77 + 98 - 76 - 90 + 63 + 82 - 72 - 92 - 81 - 96 + 74 + 94 - 65 - 86 + 74 + 88) =$$

$$= \quad 4/16 = \quad 0.25$$

$$b_{GT} = \frac{1}{N}(X_{GT})^T y = \frac{1}{16}(+77 + 98 + 76 + 90 - 63 - 82 - 72 - 92 - 81 - 96 - 74 - 94 + 65 + 86 + 74 + 88) =$$

$$= 48/16 = \quad 3$$

$$b_{LGT} = \frac{1}{N}(X_{LGT})^T y = \frac{1}{16}(+77 + 98 + 76 + 90 - 63 - 82 - 72 - 92 - 81 - 96 - 74 - 94 + 65 + 86 + 74 + 88)$$

Full Factorial Experiments

We further have to calculate the residual variance = mean square error:

$$s^2 = \frac{\mathrm{RSS}}{N - \mathrm{rank}(\boldsymbol{X})} = \frac{\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}}{N - \mathrm{rank}(\boldsymbol{X})} = \frac{(\boldsymbol{y} - \hat{\boldsymbol{y}})^{\mathrm{T}}(\boldsymbol{y} - \hat{\boldsymbol{y}})}{N - \mathrm{rank}(\boldsymbol{X})}$$

where

$$N = 2^n \times K$$

and it is easy to see that

$$\mathrm{rank}(\boldsymbol{X}) = 2^n$$

Moreover, the predicted values are

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}\boldsymbol{C}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y} = \frac{1}{N}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y}$$

Full Factorial Experiments

Calculating, we obtain that:

$$XX^{\mathrm{T}} = \begin{pmatrix} 2^n\boldsymbol{E} & & & \\ & 2^n\boldsymbol{E} & & \\ & & \ddots & \\ & & & 2^n\boldsymbol{E} \end{pmatrix}$$

i.e. the $(2^n \times K) \times (2^n \times K)$ matrix

with the $K \times K$ matrix $2^n\boldsymbol{E}$ repeated 2^n-times on its diagonal.

The symbol

$\boldsymbol{E}$ is the $K \times K$ matrix of all ones

and

$2^n\boldsymbol{E}$ is the $K \times K$ matrix of the numbers 2^n

Full Factorial Experiments

Substituting into the above equation

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}\boldsymbol{C}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y} = \frac{1}{N}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y} = \frac{1}{2^n \times K}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{y}$$

and calculating, we obtain the predicted values:

$$\hat{y}_{sk} = \frac{1}{K}(y_{s1} + y_{s2} + \cdots + y_{sK}) \qquad \text{for} \quad s \in \mathcal{S} \qquad \text{for} \quad k = 1, 2, \ldots, K$$

where – recall –

$$\mathcal{S} = \{\pm\}^n = \{+, -\}^n$$

is the index set of all the 2^n possible combinations of the levels of the factors and K is the number of the replications of the experiment for the given

Full Factorial Experiments

We have:

$$s^2 = \frac{\mathrm{RSS}}{N - \mathrm{rank}(X)} = \frac{e^{\mathrm{T}} e}{N - \mathrm{rank}(X)} = \frac{(y - \hat{y})^{\mathrm{T}}(y - \hat{y})}{N - \mathrm{rank}(X)} = \frac{(y - \hat{y})^{\mathrm{T}}(y - \hat{y})}{2^n \times K - 2^n}$$

Substituting, we have:

$$s^2 = \frac{(y - \hat{y})^{\mathrm{T}}(y - \hat{y})}{(K-1)2^n} = \frac{1}{(K-1)2^n} \sum_{s \in S} \sum_{k=1}^{K} (y_{sk} - \hat{y}_{sk})^2 =$$

$$= \frac{1}{(K-1)2^n} \sum_{s \in S} \sum_{k=1}^{K} \left(y_{sk} - \frac{y_{s1} + y_{s2} + \cdots + y_{sK}}{K} \right)^2$$

Full Factorial Experiments: Example

In our example, we have:

Factor	L	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+	+
	G	−	−	−	−	+	+	+	+	−	−	−	−	+	+	+	+
	T	−	−	−	−	−	−	−	−	+	+	+	+	+	+	+	+
Observed	$y_{\pm\pm\pm k}$	77	81	98	96	76	74	90	94	63	65	82	86	72	74	92	88
Sample means	$\hat{y}_{\pm\pm\pm}$	79		97		75		92		64		84		73		90	

$$
\begin{aligned}
\text{RSS} = {} & (77-79)^2 + (81-79)^2 + (98-97)^2 + (96-97)^2 + (76-75)^2 + \\
& + (74-75)^2 + (90-92)^2 + (94-92)^2 + (63-64)^2 + (65-64)^2 + \\
& + (82-84)^2 + (86-84)^2 + (72-73)^2 + (74-73)^2 + (92-90)^2 + \\
& + (88-90)^2 \quad = \quad 40
\end{aligned}
$$

and

$$s^2 = \frac{\text{RSS}}{(K-1)2^n} = \frac{40}{(2-1)\times 2^3} = \frac{40}{8} = 5$$

Full Factorial Experiments

Finally, we calculate the statistic

$$T = \frac{b_j}{\sqrt{s^2}\sqrt{c_{jj}}} =$$

$$= \frac{\frac{1}{N}(X_j)^{\mathrm{T}} y}{\sqrt{\frac{\mathrm{RSS}}{(K-1)2^n}}\sqrt{\frac{1}{N}}} = \sqrt{\frac{1}{K2^n}}(X_j)^{\mathrm{T}} y \sqrt{\frac{(K-1)2^n}{\mathrm{RSS}}} = \sqrt{\frac{(K-1)/K}{\mathrm{RSS}}}(X_j)^{\mathrm{T}} y$$

Full Factorial Experiments

Finally, we perform the *t*-test of the significance of the factor or the interaction.

To this end, we compare the value of the above statistic T with the quantile

$$t_{N-2^n}\left(1-\frac{\alpha}{2}\right)$$

- If $|T| \geq t_{N-2^n}(1-\alpha/2)$, the critical region,

  then **reject** the null hypothesis that $b_j = 0$,

  that is, consider the factor / interaction significant.

- If $|T| < t_{N-2^n}(1-\alpha/2)$, then **fall to reject** the null hypothesis $b_j = 0$,

  that is, consider the factor / interaction insignificant – **neglect it**.

Full Factorial Experiments: Example

In our example – choosing significance level $\alpha = 5\ \%$ – we have the quantile

$$q = t_{N-2^n}\left(1-\frac{\alpha}{2}\right) = t_{16-8}\left(1-\frac{0.05}{2}\right) = t_8(0.975) \doteq 2.306004$$

and

b_{L}:

$$T = \frac{b_{\mathrm{L}}}{\sqrt{s^2}\sqrt{c_{\mathrm{LL}}}} = \frac{9}{\sqrt{5}\sqrt{1/16}} \doteq 16.100 \quad \geq q \qquad \Longrightarrow \quad \text{significant}$$

b_{G}:

$$T = \frac{b_{\mathrm{G}}}{\sqrt{s^2}\sqrt{c_{\mathrm{GG}}}} = \frac{0.75}{\sqrt{5}\sqrt{1/16}} \doteq 1.342 \quad < q \qquad \Longrightarrow \quad \text{neglect}$$

Full Factorial Experiments: Example

b_{T}:

$$T = \frac{b_{\mathrm{T}}}{\sqrt{s^2}\sqrt{c_{\mathrm{TT}}}} = \frac{-4}{\sqrt{5}\sqrt{1/16}} \doteq -7.155 \quad \leq -q \qquad \Rightarrow \quad \text{significant}$$

b_{LG}:

$$T = \frac{b_{\mathrm{LG}}}{\sqrt{s^2}\sqrt{c_{\mathrm{LGLG}}}} = \frac{-0.5}{\sqrt{5}\sqrt{1/16}} \doteq -0.894 \quad > -q \qquad \Rightarrow \quad \text{neglect}$$

b_{LT}:

$$T = \frac{b_{\mathrm{LT}}}{\sqrt{s^2}\sqrt{c_{\mathrm{LTLT}}}} = \frac{0.25}{\sqrt{5}\sqrt{1/16}} \doteq 0.447 \quad < q \qquad \Rightarrow \quad \text{neglect}$$

b_{GT}:

$$T = \frac{b_{\mathrm{GT}}}{\sqrt{s^2}\sqrt{c_{\mathrm{GTGT}}}} = \frac{3}{\sqrt{5}\sqrt{1/16}} \doteq 5.367 \quad > q \qquad \Rightarrow \quad \text{significant}$$

Full Factorial Experiments: Example

b_{LGT}:

$$T = \frac{b_{\mathrm{LGT}}}{\sqrt{s^2}\sqrt{c_{\mathrm{LGTLGT}}}} = \frac{-0.25}{\sqrt{5}\sqrt{1/16}} \doteq -0.447 \;\; > -q \qquad \Longrightarrow \;\; \text{neglect}$$

CONCLUSION:

the factors L T and the interaction GT

have significant effect on the lifespan of the spring.

Moreover, we conclude that

$$Y \approx b_0 + x_{\mathrm{L}} b_{\mathrm{L}} + x_{\mathrm{T}} b_{\mathrm{T}} + x_{\mathrm{GT}} b_{\mathrm{GT}}$$

$$\approx 81.75 + 9x_{\mathrm{L}} - 4x_{\mathrm{T}} + 3x_{\mathrm{GT}} \qquad \text{where} \quad x_{\mathrm{L}}, x_{\mathrm{T}}, x_{\mathrm{GT}} \in \{-1, +1\}$$

Graphical assessment of factor / interaction significance

Full Factorial Experiments

If the number of the replications of each experiment is

$$K = 1$$

then the above computational procedure cannot be used

because

$$s^2 = \frac{\mathrm{RSS}}{(K-1)2^n}$$

that is, we cannot perform the calculations because we would divide by zero,

or we would always calculate $T = 0$.

We perform the graphical assessment of the significance then.

Full Factorial Experiments

Assuming $K = 1$, so that $N = 2^n$,

the graphical assessment proceeds as follows:

- Sort all the estimates b_s, for $s \in \mathcal{S}$, in ascending order.
- Denote the i-th least value by $b_{(i)}$ for $i = 1, 2, \ldots, 2^n - 1$ so that we have

$$b_{(1)} \leq b_{(2)} \leq \cdots \leq b_{(2^n-1)}$$

- Calculate the values

$$P_i = \frac{i - \frac{1}{2}}{2^n - 1} \qquad \text{for} \quad i = 1, 2, \ldots, 2^n - 1$$

- Plot the points

$$\left[b_{(i)}, P_i\right] \qquad \text{for} \quad i = 1, 2, \ldots, 2^n - 1$$

Full Factorial Experiments: Example

In our example, we have:

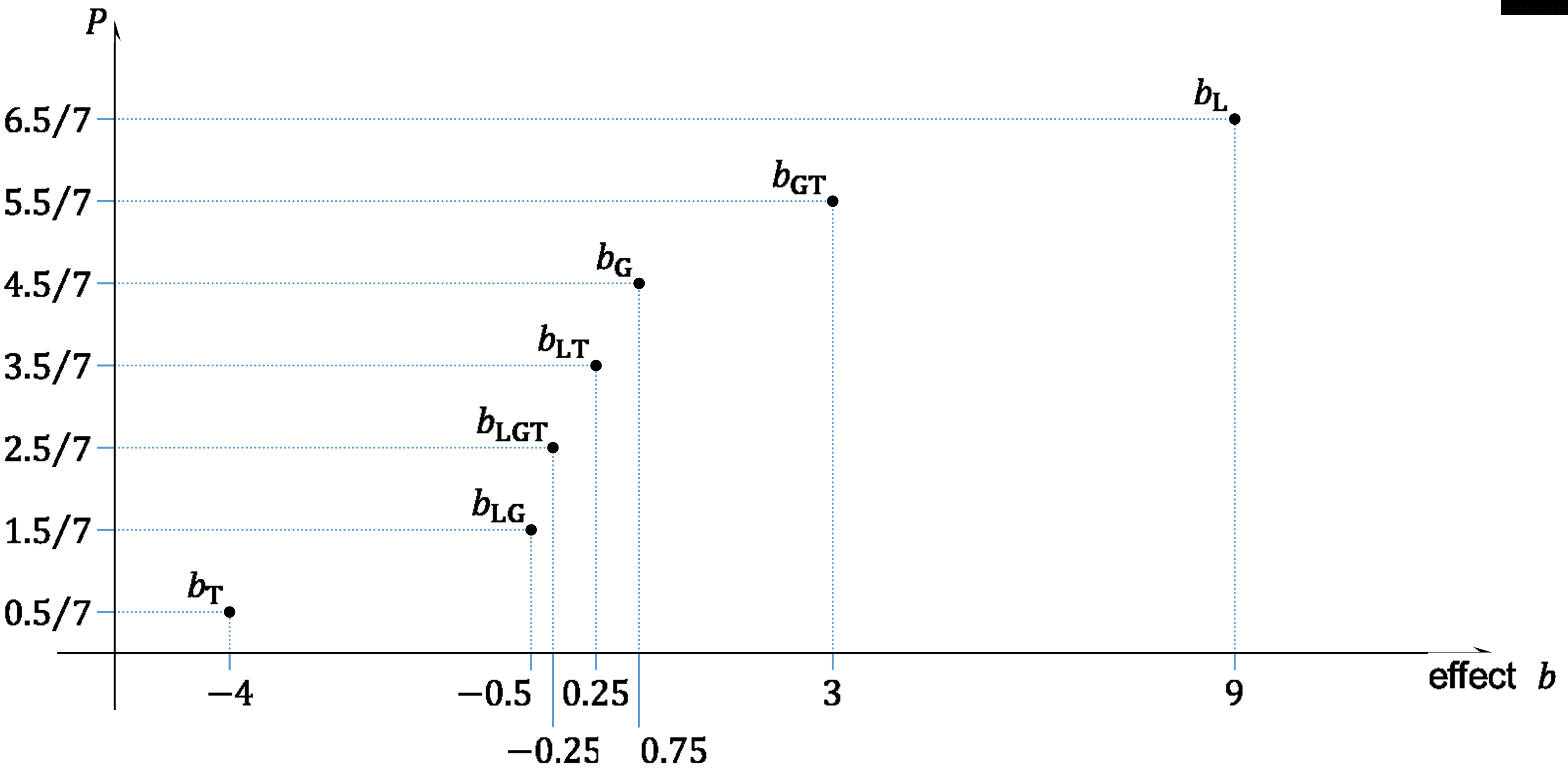
$$b_{\mathrm{T}} < b_{\mathrm{LG}} < b_{\mathrm{LGT}} < b_{\mathrm{LT}} < b_{\mathrm{G}} < b_{\mathrm{GT}} < b_{\mathrm{L}}$$

$$-4 < -0.5 < -0.25 < 0.25 < 0.75 < 3 < 9$$

and P_i:

$$\frac{0.5}{7} < \frac{1.5}{7} < \frac{2.5}{7} < \frac{3.5}{7} < \frac{4.5}{7} < \frac{5.5}{7} < \frac{6.5}{7}$$

Full Factorial Experiments: Example



Full Factorial Experiments: Graphical assessment

The points $[b_{(i)}, P_i]$ take the shape of an S-curve sometimes.

Those points that lie out of the approximately linear middle part of the S-curve indicate the significant factors or interactions.

<u>In our example:</u> the points

$$b_{\mathrm{T}} \qquad b_{\mathrm{GT}} \qquad b_{\mathrm{L}}$$

lie out of the approximately linear middle part of the S-curve,

which is the same result that we obtained by using the *t*-test above.

Graphs of interaction effects

Full Factorial Experiments: Interaction effect

There is an **interaction effect** between two factors in a full factorial experiment when the effect of one independent variable (factor) depends on the level of another independent variable (factor).

Considering two factors A and B, take

- all the observations when factors A and B were at the levels $--$, respectively
- all the observations when factors A and B were at the levels $-+$, respectively
- all the observations when factors A and B were at the levels $+-$, respectively
- all the observations when factors A and B were at the levels $++$, respectively

and calculate the <u>average</u> (<u>sample mean</u>) for each of the four groups

Full Factorial Experiments: Interaction effect

We then have:

Factor B	+	$\bar{y}_{-+}$	$\bar{y}_{++}$
	−	$\bar{y}_{--}$	$\bar{y}_{+-}$
		−	+
		Factor A	

We then depict the values in the form of a chart

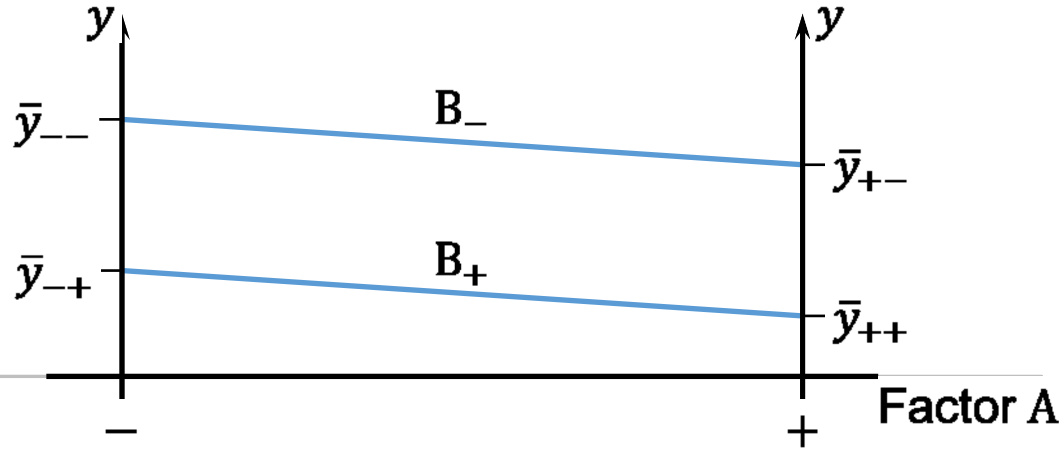
(here as the dependence of Factor B on Factor A):

- If the lines are ≈ parallel,

  then the interaction is not significant.

- If the lines are not parallel,

  then the interaction is significant.



Full Factorial Experiments: Example

In our example, we have:

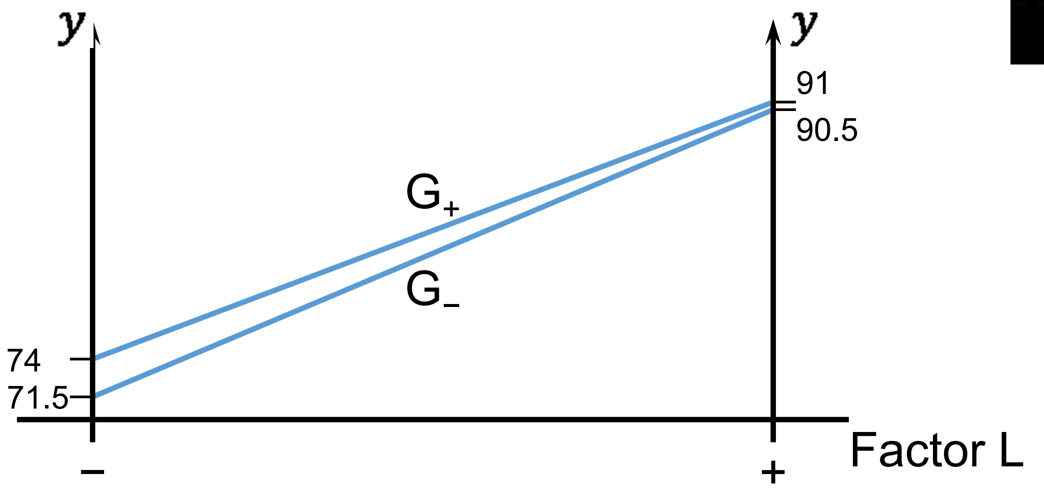
Factor	L	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+	+
	G	−	−	−	−	+	+	+	+	−	−	−	−	+	+	+	+
	T	−	−	−	−	−	−	−	−	+	+	+	+	+	+	+	+
Observed	$y_{\pm\pm\pm k}$	77	81	98	96	76	74	90	94	63	65	82	86	72	74	92	88

Factor G	+	76,□ 74,□ 72,□ 7	90,□ 94,□ 92,□ 8
		74	**91**
	−	77,□ 81,□ 63,□ 6	98,□ 96,□ 82,□ 8
		71.5	**90.5**
		−	+
		Factor L	

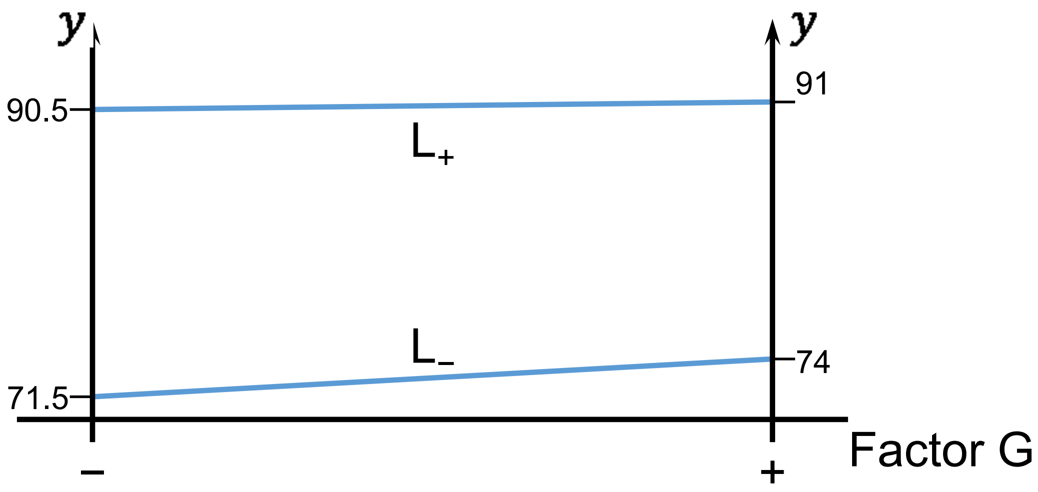
Factor T	+	63,□ 65,□ 72,□ 7	82,□ 86,□ 92,□ 8
		68.5	**87**
	−	77,□ 81,□ 76,□ 7	98,□ 96,□ 90,□ 9
		77	**94.5**
		−	+
		Factor L	

Full Factorial Experiments: Example

G	+	74	91
	−	71.5	90.5
		−	+
		L	



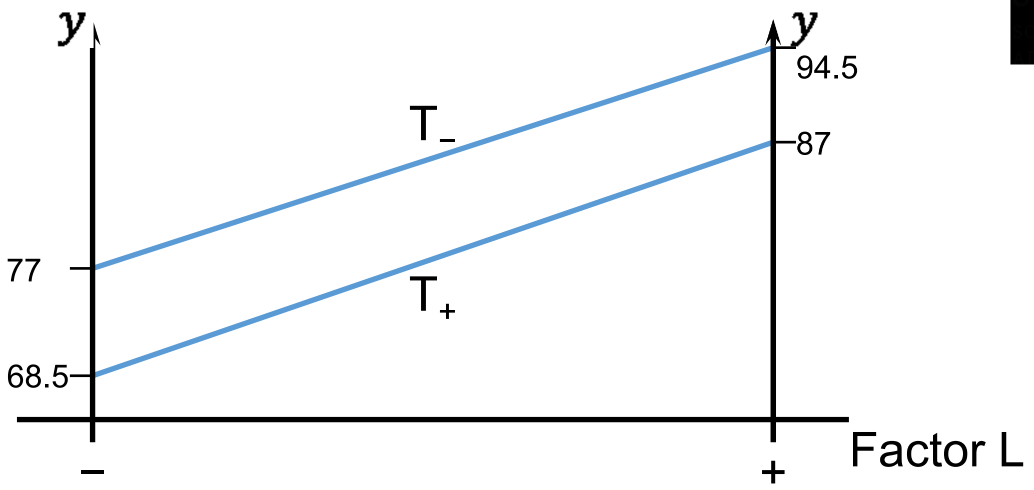
L	+	90.5	91
	−	71.5	74
		−	+
		G	



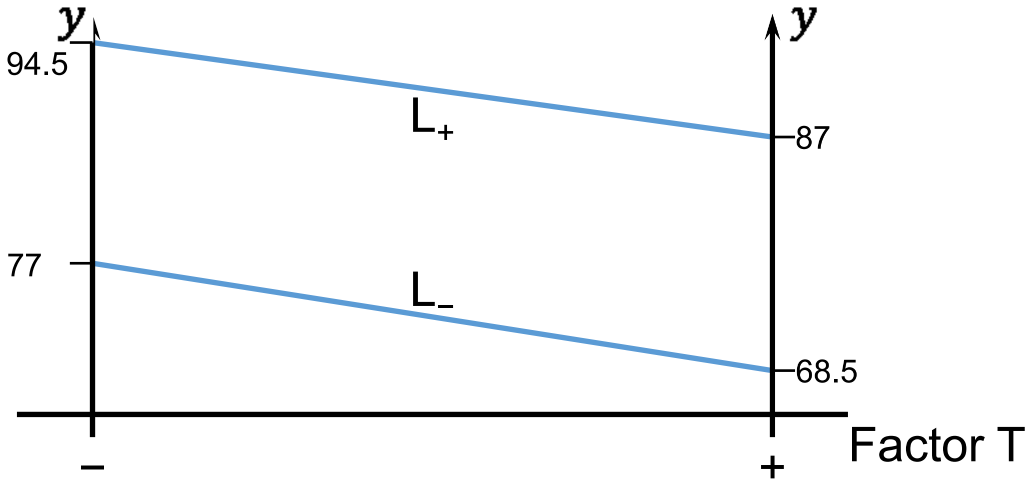
≈ parallel ⇨ the interaction is not significant

Full Factorial Experiments: Example

T	+	68.5	87
	−	77	94.5
		−	+
		L	



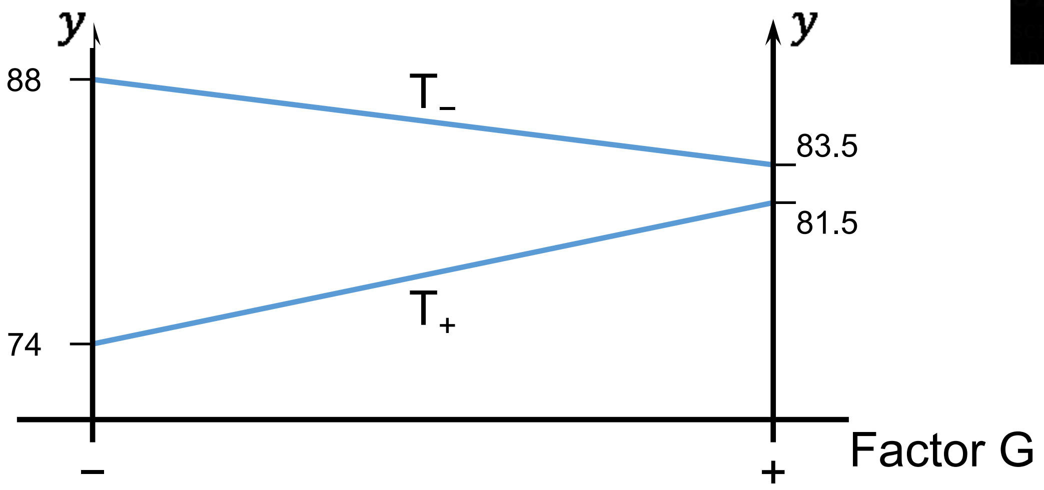
L	+	94.5	87
	−	77	68.5
		−	+
		T	



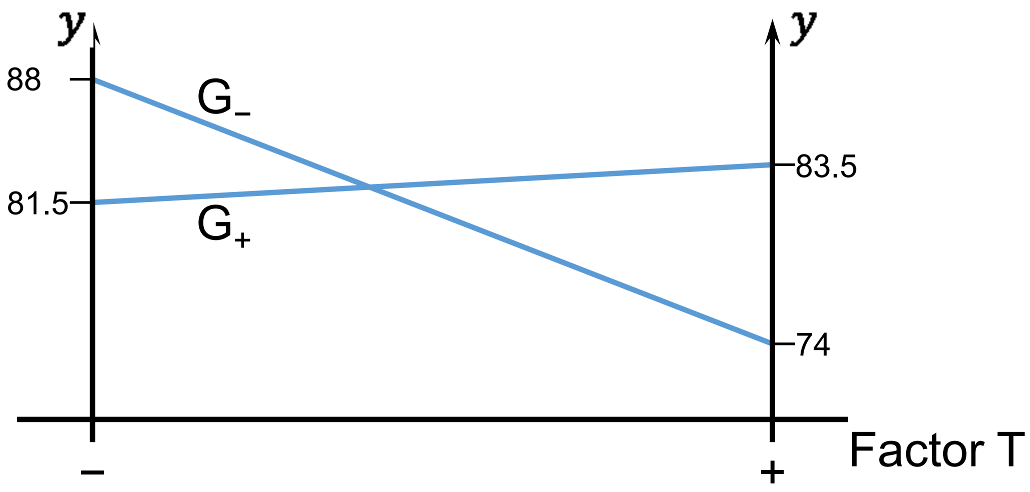
≈ parallel ⇨ the interaction is not significant

Full Factorial Experiments: Example

		G −	G +
T	+	74	83.5
T	−	88	81.5



		T −	T +
G	+	81.5	83.5
G	−	88	74



NOT parallel ⇨ the interaction is SIGNIFICANT