Quantitative Methods

Lecture 3

Matrix calculus

SILESIAN UNIVERSITY
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Matrix calculus

Vector space of matrices

(addition and scalar multiplication)

Rank of a matrix

Multiplication of matrices

Square matrices

Singular and non-singular matrices

Matrix

Let $m, n \in \mathbb{N}$ be natural numbers.

Then a real $m \times n$ **matrix** (or a matrix of the type $m \times n$ or an m-by-n matrix) is a rectangular table of real numbers consisting of m rows and of n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The elements a_{ij} are the **entries** of the matrix A
for $i = 1, 2, \ldots, m$ and for $j = 1,2, \ldots, n$.

The vector space of matrices

Let $m, n \in \mathbb{N}$ be natural numbers.

The space of all matrices of the type $m \times n$ is denoted by $\mathbb{R}^{m \times n}$.

On the space $\mathbb{R}^{m \times n}$ of the matrices of the type $m \times n$,
the following vector operations are introduced:

— addition

— subtraction

— scalar multiplication

Addition of matrices

Let $m, n \in \mathbb{N}$ be natural numbers. Let two matrices $A, B \in \mathbb{R}^{m \times n}$ be given.

Addition:

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Subtraction of matrices

Let $m, n \in \mathbb{N}$ be natural numbers. Let two matrices $A, B \in \mathbb{R}^{m \times n}$ be given.

Subtraction:

$$A - B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{pmatrix}$$

Multiplication of a matrix by a scalar

Let $m, n \in \mathbb{N}$ be natural numbers.
Let a matrix $A \in \mathbb{R}^{m \times n}$ and a scalar $\lambda \in \mathbb{R}$ be given.

Scalar multiplication:

$$\lambda A = \lambda \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}$$

The zero matrix

Let $m, n \in \mathbb{N}$ be natural numbers.

The **zero matrix** of the type $m \times n$ is

$$O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The zero matrix consists of the zeros (0) only.

The vector space of matrices

Let $m, n \in \mathbb{N}$ be natural numbers.

For any matrices $A, B, C \in \mathbb{R}^{m \times n}$ and for any scalars $\lambda, \mu \in \mathbb{R}$, it holds:

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + O = A = O + A$$

$$A - A = O$$

$$(\lambda + \mu)A = \lambda A + \mu A$$

$$\lambda(A + B) = \lambda A + \lambda B$$

$$(\lambda\mu)A = \lambda(\mu A)$$

$$1A = A$$

We can see now that the space $\mathbb{R}^{m \times n}$ of the matrices of the type $m \times n$ is a real vector space of dimension $m \times n$.

The rank of a matrix

Let $m, n \in \mathbb{N}$ be natural numbers.

The **rank** of a real $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is

= the number of linearly independent columns

= the number of linearly independent rows

Observation: The $\operatorname{rank}(A) \leq \min(m, n)$

Multiplication of matrices

Let $m, n, p \in \mathbb{N}$ be natural numbers.

Given a matrix $A \in \mathbb{R}^{m \times n}$ and a matrix $B \in \mathbb{R}^{n \times p}$, we can **multiply** them:

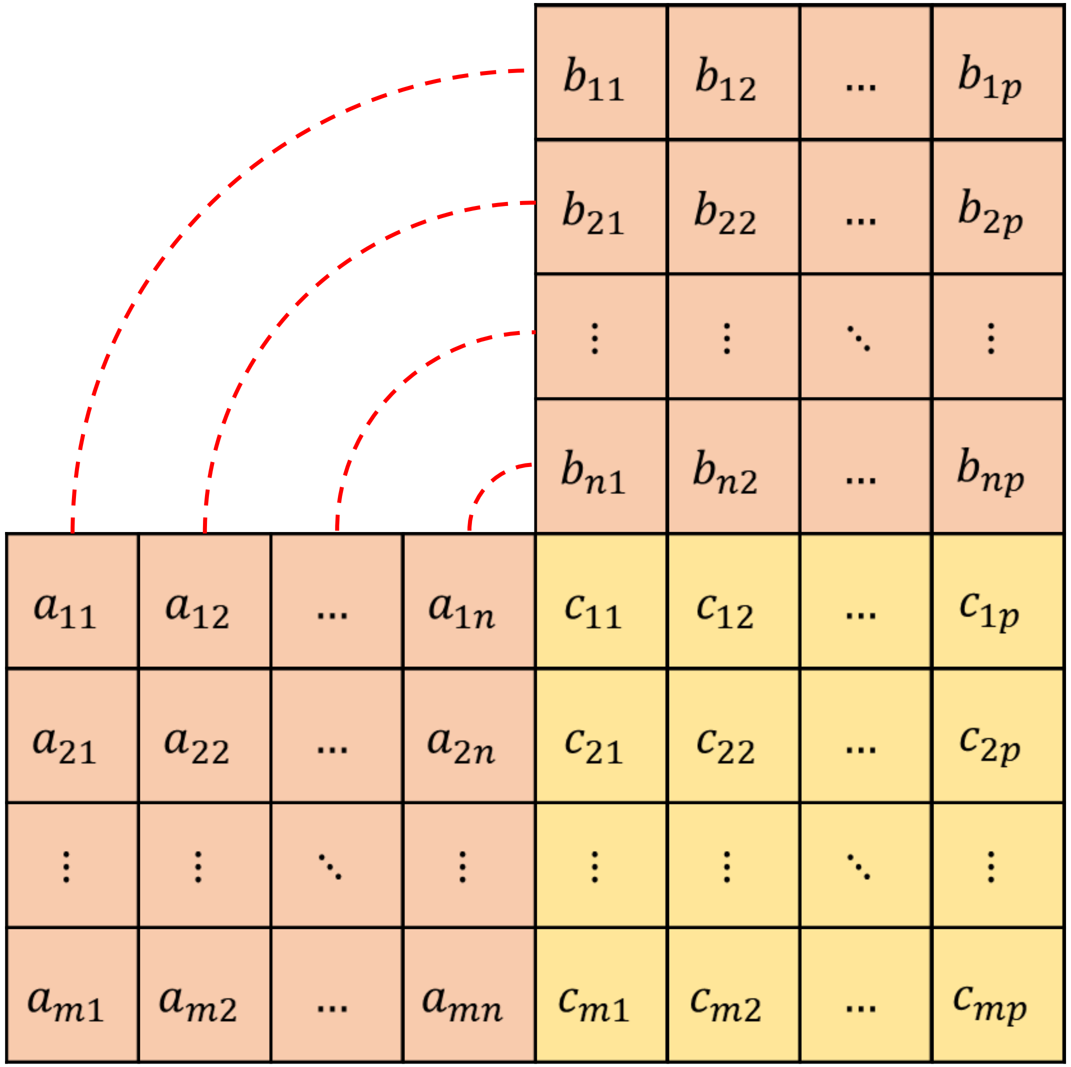
$$C = AB$$

The entries c_{ik} of the **product** $C \in \mathbb{R}^{m \times p}$ are calculated as follows:

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$$

for $i = 1, 2, \ldots, m$ and for $k = 1, 2, \ldots, p$.

Multiplication of matrices



$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$$

for $i = 1, 2, \ldots, m$ and
for $k = 1, 2, \ldots, p$

Square matrices

Let $n \in \mathbb{N}$ be a natural number.

A **square matrix** is any matrix of type $n \times n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The identity matrix

Let $n \in \mathbb{N}$ be a natural number.

The **identity matrix** of the type $n \times n$ is the matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

The identity matrix has

— ones (1) on its main diagonal and

— zeros (0) everywhere else.

Non-singular square matrices

Let $n \in \mathbb{N}$ be a natural number.

A square matrix $A \in \mathbb{R}^{n \times n}$ is

- either singular,
- or non-singular.

The matrix A is **non-singular** if and only if there exists a square matrix $B \in \mathbb{R}^{n \times n}$ such that

$$AB = I = BA$$

where I is the identity matrix.

Notice that there exists no more than one such a matrix B.

The inverse matrix

Let $n \in \mathbb{N}$ be a natural number and let $A \in \mathbb{R}^{n \times n}$ be a non-singular square matrix.

Then we already know that there exists exactly one square matrix $B \in \mathbb{R}^{n \times n}$ such that

$$AB = I = BA$$

where I is the identity matrix.

The matrix B is the **matrix inverse** to the matrix A and it is denoted by

$$B = A^{-1}$$

Singular and non-singular matrices

Let $n \in \mathbb{N}$ be a natural number and let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

We know that the matrix A is non-singular if and only if there exists exactly one square matrix $B \in \mathbb{R}^{n \times n}$ such that

$$AB = I = BA$$

Theorem. The square matrix A is non-singular if and only if $\mathrm{rank}(A) = n$, i.e.

— all n rows of the matrix are linearly independent
— all n columns of the matrix are linearly independent

The matrix is **singular** if and only if it is *not non-singular*.

Summary I

Let $n \in \mathbb{N}$ be a natural number, let $A, B, C \in \mathbb{R}^{n \times n}$ be square matrices, and let $\lambda, \mu \in \mathbb{R}$ be scalars. It holds:

$$A + B = B + A$$

$$(\lambda + \mu)A = \lambda A + \mu A$$

$$(A + B) + C = A + (B + C)$$

$$\lambda(A + B) = \lambda A + \lambda B$$

$$A + O = A = O + A$$

$$(\lambda\mu)A = \lambda(\mu A)$$

$$A - A = O$$

$$(AB)C = A(BC)$$

$$1A = A$$

$$AI = A = IA$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

Summary II

Let $n \in \mathbb{N}$ be a natural number and let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

If, moreover, the matrix A is non-singular, then it also holds

$$AA^{-1} = I = A^{-1}A$$