Quantitative Methods

Lecture 5

Sequences, series
and infinite sums

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BAKVM

Outline of the lecture

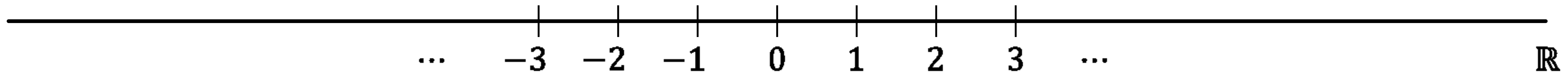
- Sequence of real numbers
- Arithmetic progression and Geometric progression
- The sum of the first n elements of the arithmetic or geometric progression
- The limit of a sequence and its properties
- Series (infinite sums)
- The sum of a geometric series
- Criteria of convergence, alternating series, Leibniz criterion

A sequence of real numbers

Recall the set of the **natural** numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$$

and the set of the **real** numbers:



We shall now deal with sequences of real numbers:

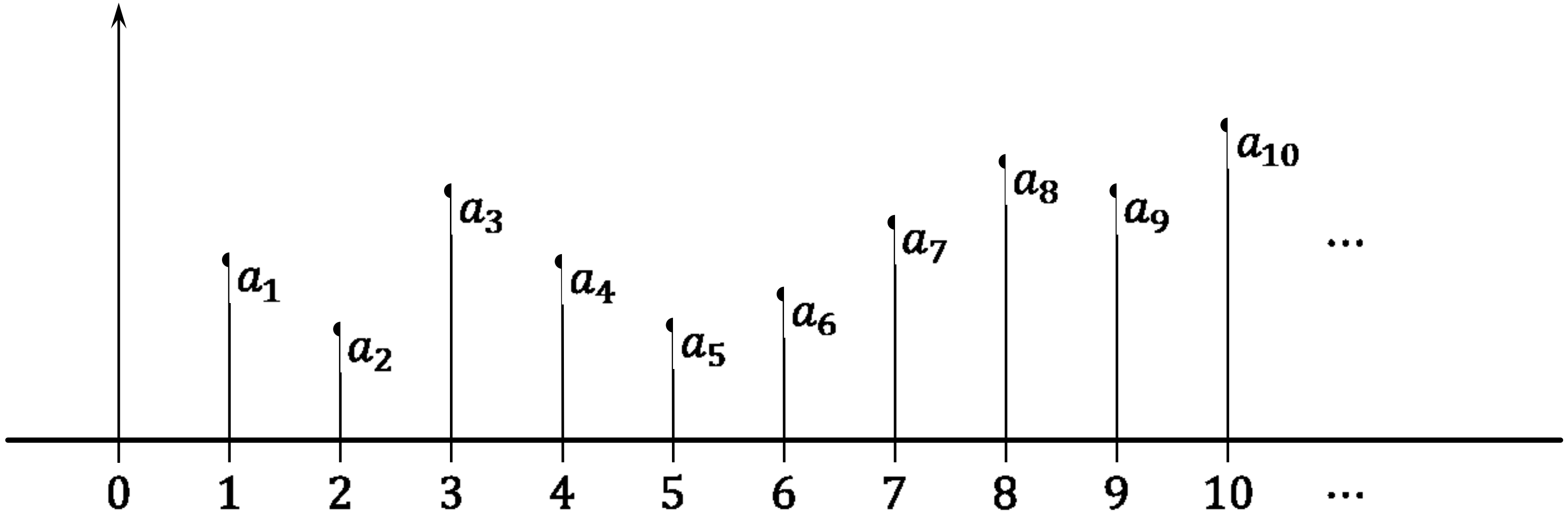
$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, \dots \in \mathbb{R}$$

A sequence of real numbers

Recall that $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots\}$.

A sequence of real numbers $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, \dots \in \mathbb{R})$
is a mapping, or function,

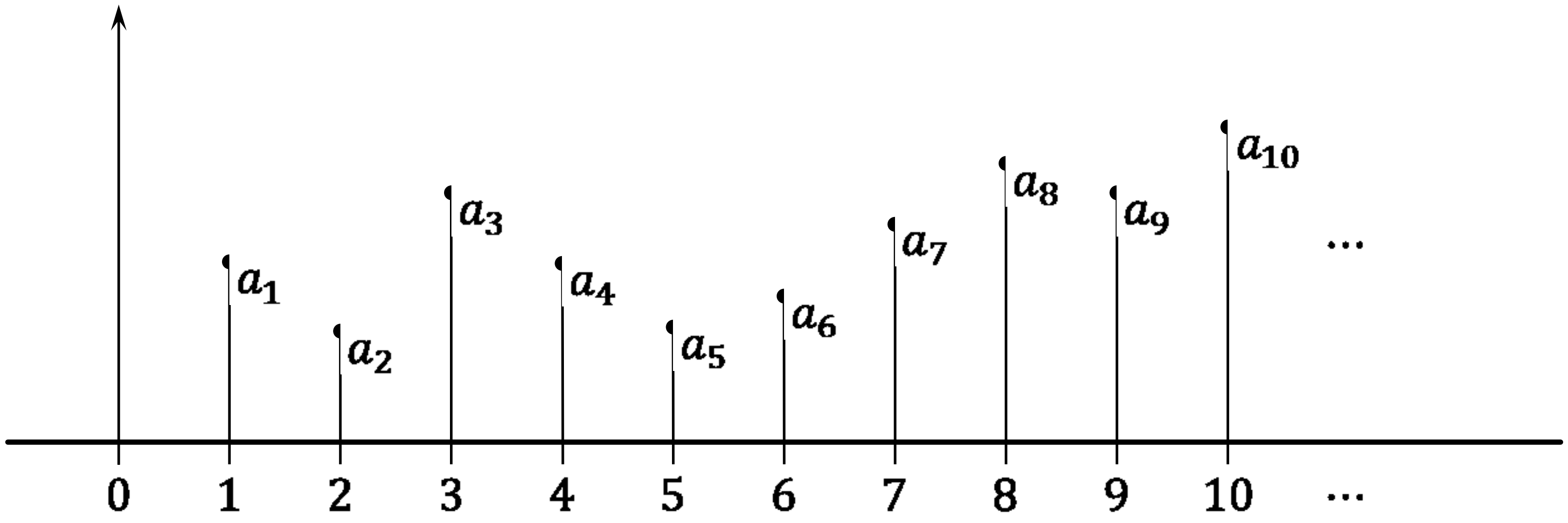
$$a: \mathbb{N} \longrightarrow \mathbb{R}$$
$$a: n \longmapsto a_n \qquad \text{for} \quad n \in \mathbb{N}$$



A sequence of real numbers

The sequence of real numbers $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, \ldots \in \mathbb{R})$,
i.e. the mapping $a\colon \mathbb{N} \longrightarrow \mathbb{R}$, is denoted by

$$\{a_n\}_{n=1}^{\infty}$$



Arithmetic progression

The sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is an **arithmetic progression**

if and only if the difference

$$d = a_{n+1} - a_n$$

is constant for all $n = 1, 2, 3, \ldots$

The n-th element of the arithmetic progression is

$$a_n = a_1 + (n-1)d \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

Arithmetic progression: the sum of the first *n* elements

Let the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers be an **arithmetic progression.**

The sum of the first n elements is

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and it holds

$$\sum_{k=1}^{n} a_k = \frac{a_1 + a_n}{2}$$

Geometric progression

The sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a **geometric progression**

if and only if the ratio

$$q = \frac{a_{n+1}}{a_n}$$

is constant for all $n = 1, 2, 3, \ldots$

The n-th element of the geometric progression is

$$a_n = a_1 \times q^{n-1} \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

Geometric progression: the sum of the first *n* elements

Let the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers be a **geometric progression,** i.e. $a_n = a_1 \times q^{n-1}$ for some $q \in \mathbb{R}$.

Notice that:

$$a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n =$$
$$= a_1 \times q^0 + a_1 \times q^1 + a_1 \times q^2 + a_1 \times q^3 + a_1 \times q^4 + \cdots + a_1 \times q^{n-1} =$$
$$= a_1 \times (q^0 + q^1 + q^2 + q^3 + q^4 + \cdots + q^{n-1}) = S$$

Hence

$$a_1 \times (q^0 + q^1 + q^2 + q^3 + q^4 + \cdots + q^{n-1})(q-1) = S(q-1)$$
$$a_1 \times (q^1 - q^0 + q^2 - q^1 + q^3 - q^2 + \cdots + q^n - q^{n-1}) = S(q-1)$$
$$a_1 \times (q^n - q^0) = S(q-1)$$
$$S = a_1 \times \frac{q^n - 1}{q - 1}$$

Geometric progression: the sum of the first *n* elements

Let the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers be a **geometric progression.**

The sum of the first n elements is

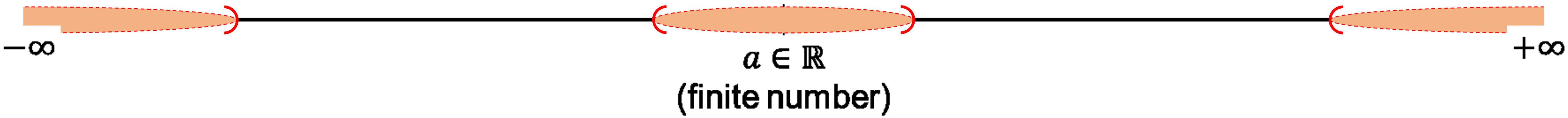
$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and it holds

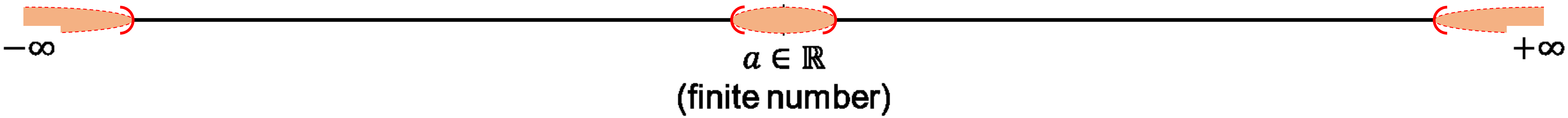
$$\sum_{k=1}^{n} a_k = a_1 \times \frac{q^n - 1}{q - 1}$$

The limit of a sequence: the neighbourhood of a point

The **neighbourhood** of a point is the set of all “nearby” points:



A **refined neighbourhood** of the point is a smaller set of all “nearby” points:



And we are about to consider smaller and smaller neighbourhoods…

The limit of a sequence

Consider a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.

It may happen that:

as n tends to infinity $(n \to +\infty)$,
then the numbers a_n tend to some number A, i.e. $a_n \to A$.

In other words:

as n gets close to infinity (the point $+\infty$),
then the numbers a_n are close to the number A.

Equivalently:

if n is in a small neighbourhood of infinity (the point $+\infty$),
then the numbers a_n are in a small neighbourhood of the number A.

The limit of a sequence

To denote the fact that the number A is the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, we write

$$\lim_{n\to\infty} a_n = A$$

Notice that the number A can be

— either finite ($A \in \mathbb{R}$)

— or infinite ($A = -\infty$ or $A = +\infty$)

Hence, we need up to three definitions of the notation "$\lim_{n\to\infty} a_n = A$" (for three types of neighbourhood of the point A; notice that $n \to \infty$, so the neighbourhood of $+\infty$ is the same in all of the three cases).

The limit of a sequence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers and
let $A \in \mathbb{R}$ be a (finite) real number.

We say that the number A is the **limit of the sequence** $\{a_n\}_{n=1}^{\infty}$ and we write

$$\lim_{n\to\infty} a_n = A$$

if and only if

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \in \mathbb{N}: \ n \in (n_0, +\infty) \Rightarrow a_n \in (A - \varepsilon, A + \varepsilon)$$

or

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \in \mathbb{N}: \ n > n_0 \Rightarrow |a_n - A| < \varepsilon$$

The limit of a sequence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

We say that the (infinite) number $+\infty$ is the **limit** of the sequence $\{a_n\}_{n=1}^{\infty}$ and we write

$$\lim_{n\to\infty} a_n = +\infty$$

if and only if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}:\ n \in (n_0, +\infty) \Longrightarrow a_n \in (K, +\infty)$$

or

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}:\ n > n_0 \Longrightarrow a_n > K$$

The limit of a sequence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

We say that the (infinite) number $-\infty$ is the **limit** of the sequence $\{a_n\}_{n=1}^{\infty}$ and we write

$$\lim_{n\to\infty} a_n = -\infty$$

if and only if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}:\ n \in (n_0, +\infty) \Rightarrow a_n \in (-\infty, K)$$

or

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}:\ n > n_0 \Rightarrow a_n < K$$

The limit of a sequence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

We say that the sequence $\{a_n\}_{n=1}^{\infty}$ is **convergent** if and only if $\lim_{n\to\infty} a_n = A$ for some *finite* number $A \in \mathbb{R}$.

The sequence $\{a_n\}_{n=1}^{\infty}$ **diverges to** $+\infty$ if and only if $\lim_{n\to\infty} a_n = +\infty$.

The sequence $\{a_n\}_{n=1}^{\infty}$ **diverges to** $-\infty$ if and only if $\lim_{n\to\infty} a_n = -\infty$.

We say that the sequence $\{a_n\}_{n=1}^{\infty}$ is **divergent** if and only if $\lim_{n\to\infty} a_n = A$ for *no* number $A \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Properties of the limit

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that

$$\lim_{n\to\infty} a_n = A \qquad \text{and} \qquad \lim_{n\to\infty} b_n = B.$$

It then holds for any $c \in \mathbb{R}$ that:

- $\lim_{n\to\infty} (c a_n) = cA$
- $\lim_{n\to\infty} (a_n + b_n) = A + B$
- $\lim_{n\to\infty} (a_n - b_n) = A - B$

whenever the expression on the right-hand side is defined.

We define for any $a \in \mathbb{R}$:

$$a + (+\infty) = +\infty + a = +\infty$$
$$a + (-\infty) = -\infty + a = -\infty$$
$$(+\infty) + (+\infty) = +\infty$$
$$(-\infty) + (-\infty) = -\infty$$

The expressions

$$(+\infty) + (-\infty)$$
$$(+\infty) - (+\infty)$$
$$(-\infty) - (-\infty)$$

are not defined.

Properties of the limit

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that

$$\lim_{n\to\infty} a_n = A \qquad \text{and} \qquad \lim_{n\to\infty} b_n = B.$$

It then holds:

— $\lim_{n\to\infty} (a_n \times b_n) = A \times B$

— $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ if $B \neq 0$

whenever the expression on the right-hand side is defined.

We define for any $a \in \mathbb{R}^+$:

$$a \times (+\infty) = +\infty \times a = +\infty$$
$$a \times (-\infty) = -\infty \times a = -\infty$$

We define for any $a \in \mathbb{R}^-$:

$$a \times (+\infty) = +\infty \times a = -\infty$$
$$a \times (-\infty) = -\infty \times a = +\infty$$

Moreover:

$$(+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty$$
$$(+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty$$

The expressions

$$0 \times (\pm\infty)$$
$$(\pm\infty) \div (\pm\infty)$$

are not defined.

The limit as a linear mapping

Let V and W be vector spaces $f: V \to W$ be a mapping.
Recall that the mapping f is linear if and only if it holds

$$f(u+v) = f(u) + f(v) \qquad \text{for every} \quad u, v \in V$$

$$f(\lambda u) = \lambda f(u) \qquad \text{for every} \quad u \in V \quad \text{and for every} \quad \lambda \in \mathbb{R}$$

It holds

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \qquad \text{for every} \quad \{a_n\}_{n=1}^{\infty} \text{ and } \{b_n\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \lambda a_n = \lambda \lim_{n\to\infty} a_n \qquad \text{for every} \quad \{a_n\}_{n=1}^{\infty} \quad \text{and for every} \quad \lambda \in \mathbb{R}$$

Theorem

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers such that

$$a_n \leq b_n \leq c_n \qquad \text{for all} \quad n \in \mathbb{N}$$

If

$$\lim_{n\to\infty} a_n = B = \lim_{n\to\infty} c_n \qquad \text{for some} \quad B \in \mathbb{R} \cup \{+\infty, -\infty\}$$

then

$$\lim_{n\to\infty} b_n = B$$

Infinite series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

The infinite **series** or the **infinite sum** is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \cdots$$

Sequence of partial sums

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

The sequence of the **partial sums** of the sequence $\{a_n\}_{n=1}^{\infty}$ is

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
s_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\
s_6 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\
s_7 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \\
&\vdots
\end{aligned}
$$

The sum of a series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $\{s_n\}_{n=1}^{\infty}$ be the sequence of the partial sums ($s_n = a_1 + \cdots + a_n$).

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges** to a (finite) number $S \in \mathbb{R}$, or **diverges to** $+\infty$ or **diverges to** $-\infty$ and we write

$$\sum_{n=1}^{\infty} a_n = S \qquad \sum_{n=1}^{\infty} a_n = +\infty \qquad \sum_{n=1}^{\infty} a_n = -\infty$$

if and only if

$$\lim_{n\to\infty} s_n = S \qquad \lim_{n\to\infty} s_n = +\infty \qquad \lim_{n\to\infty} s_n = -\infty$$

respectively.

Divergent series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $\{s_n\}_{n=1}^{\infty}$ be the sequence of the partial sums ($s_n = a_1 + \cdots + a_n$).

We say that the series $\sum_{n=1}^{\infty} a_n$ is **divergent** if and only if the sequence $\{s_n\}_{n=1}^{\infty}$ is divergent.

The sum of a geometric series

Let the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers be a **geometric progression** and let $\{s_n\}_{n=1}^{\infty}$ be the sequence of the partial sums ($s_n = a_1 + \cdots + a_n$).

Assume that $|q| < 1$.

Then

$$s_n = a_1 \times \frac{q^n - 1}{q - 1} = \frac{a_1}{q-1}\Big(\underbrace{q^n}_{\to 0} - 1\Big) = a_1 + a_2 + \cdots + a_n$$

Hence the sum of the geometric series is

$$\sum_{k=1}^{\infty} a_k = a_1 \times \frac{1}{1-q}$$